

Global and Local Multiple SLEs for $\kappa \leq 4$ and Connection Probabilities for Level Lines of GFF

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Abstract

This article pertains to the classification of multiple Schramm-Loewner evolutions (SLE). We construct the pure partition functions of multiple SLE_κ with $\kappa \in (0, 4]$ and relate them to certain extremal multiple SLE measures, thus verifying a conjecture from [BBK05, KP16]. We prove that the two approaches to construct multiple SLEs — the global, configurational construction of [KL07, Law09a] and the local, growth process construction of [BBK05, Dub07, Gra07, KP16] — agree.

The pure partition functions are closely related to crossing probabilities in critical statistical mechanics models. With explicit formulas in the special case of $\kappa = 4$, we show that these functions give the connection probabilities for the level lines of the Gaussian Free Field (GFF) with alternating boundary conditions. We also show that certain functions, known as conformal blocks, give rise to multiple SLE_4 that can be naturally coupled with the GFF with appropriate boundary conditions.

1 Introduction

Conformal invariance and critical phenomena in two-dimensional statistical physics have been active areas of research in the last few decades, both in the mathematics and physics communities. Conformal invariance can be studied in terms of correlations and interfaces in the critical models, and both approaches have been successful. This article concerns conformally invariant probability measures on curves that should describe scaling limits of interfaces in critical lattice models (with suitable boundary conditions).

For one chordal curve between two boundary points, such scaling limit results have been rigorously established for many models: Percolation [Smi01, CN07], the Loop-Erased Random Walk and the Uniform Spanning Tree [LSW04], level lines of the discrete Gaussian Free Field [SS09, SS13], and the critical Ising and FK-Ising models [CS12, CDCH⁺14]. In this case, the limiting object is a random curve known as the chordal SLE_κ (Schramm-Loewner evolution), uniquely characterized by a single parameter $\kappa \geq 0$ together with conformal invariance and a domain Markov property [Sch00]. In general, interfaces of critical lattice models converge to variants of the SLE_κ : interfaces with certain boundary condition changes to $\text{SLE}_\kappa(\rho)$ processes (proven e.g. for the critical Ising model with plus-minus-free boundary conditions [HK13]); interfaces in multiply connected domains to more general variants of SLE_κ (see e.g. [Izy16] for the Ising model, and [Zha08b] for the Loop-Erased Random Walk), and so on. Similarly, multiple interfaces should converge to several interacting SLE curves, as has also been proven for some models, see e.g. [Izy16, Wu17]. These interacting random curves cannot be classified by conformal invariance and the domain Markov property alone, but additional data is needed [BBK05, Dub07, Gra07, KL07, Law09a, KP16].

It is also natural to ask questions about the global behavior of the interfaces, such as their crossing or connection probabilities. In fact, such a crossing probability, known as Cardy's formula, was a crucial ingredient in the celebrated proof of the conformal invariance of the scaling limit of critical percolation [Smi01, CN07]. In Figure 1.1, a simulation of the critical Ising model with alternating boundary

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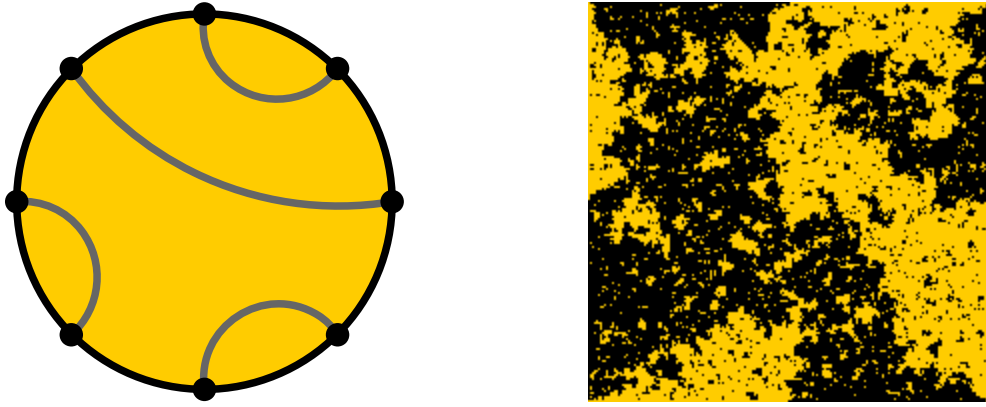


Figure 1.1: Simulation of the critical Ising model with alternating boundary conditions and the corresponding link pattern $\alpha \in \text{LP}_4$.

conditions is depicted. The figure shows one possible connectivity of the interfaces separating the black and yellow regions, but when sampling from the Gibbs measure, other planar connectivities can also arise. One may then ask with which probability do the various connectivities occur. The answer is known for Loop-Erased Random Walks ($\kappa = 2$) and for the double-dimer model ($\kappa = 4$) [KW11a, KKP17], whereas e.g. the cases of the Ising model ($\kappa = 3$) and percolation ($\kappa = 6$) are still unknown to our knowledge. In general, scaling limits of these connection probabilities should be encoded in certain quantities related to multiple SLEs, known as pure partition functions. These functions give the Radon-Nikodym derivatives of multiple SLE measures with respect to product measures of independent SLEs.

In this article, we construct the pure partition functions of multiple SLEs for all $\kappa \in (0, 4]$ and show that they are smooth, positive, and (essentially) unique. We also relate these functions to certain extremal multiple SLE measures, thus verifying a conjecture from [BBK05, KP16]. To find the pure partition functions, we give a global construction of multiple SLE_κ measures in the spirit of [KL07, Law09a, Law09b], but pertaining to the complete classification of these random curves. We also prove that, as probability measures on curve segments, these global multiple SLEs agree with another approach to construct and classify interacting SLE curves, known as local multiple SLEs [BBK05, Dub07, Gra07, KP16].

The SLE_4 processes are known to be realized as level lines of the Gaussian Free Field (GFF). In the spirit of [KW11a, KKP17], we find algebraic formulas for the pure partition functions in this case and show that they give explicitly the connection probabilities for the level lines of the GFF with alternating boundary conditions. We also show that certain functions, called conformal blocks, give rise to multiple SLE_4 processes that can be naturally coupled with the GFF with appropriate boundary conditions.

1.1 Multiple SLEs and Pure Partition Functions

One can naturally view interfaces in discrete models as dynamical processes. Indeed, in his seminal article [Sch00], O. Schramm defined the SLE_κ as a random growth process (Loewner chain) whose time evolution is encoded in an ordinary differential equation (Loewner equation, see Section 2.1). Using the same idea, one may generate processes of several SLE_κ curves by describing their time evolution via a Loewner chain. Such processes are *local multiple SLEs*: probability measures on curve segments growing from $2N$ fixed boundary points $x_1, \dots, x_{2N} \in \partial\Omega$ of a simply connected domain $\Omega \subsetneq \mathbb{C}$, only defined up to a stopping time strictly smaller than the time when the curves touch (we call this localization).

We prove in Theorem 1.3 that, when $\kappa \leq 4$, localizations of global multiple SLEs give rise to local multiple SLEs. Then, the $2N$ curve segments form N planar, non-intersecting simple curves connecting the $2N$ marked boundary points pairwise, as in Figure 1.1 for the critical Ising interfaces. Topologically, these N curves form a planar pair partition, which we call a *link pattern* and denote by

$\alpha = \{\{a_1, b_1\}, \dots, \{a_N, b_N\}\}$, where $\{a, b\}$ are the pairs in α , called *links*. The set of link patterns of N links on $\{1, \dots, 2N\}$ is denoted by LP_N . The number of elements in LP_N is the Catalan number, $\#\text{LP}_N = C_N = \frac{1}{N+1} \binom{2N}{N}$. We also denote by $\text{LP} = \bigsqcup_{N \geq 0} \text{LP}_N$ the set of link patterns of any number of links, where we include the empty link pattern $\emptyset \in \text{LP}_0$ in the case $N = 0$.

By the results of [Dub07, KP16], the local $N\text{-SLE}_\kappa$ probability measures are classified by smooth functions \mathcal{Z} of the marked points, called partition functions. It is believed that they form a C_N -dimensional space, with basis given by certain special elements \mathcal{Z}_α , called pure partition functions, indexed by the C_N link patterns $\alpha \in \text{LP}_N$. When normalized, these functions can be interpreted as (conjectured) scaling limits of crossing probabilities in discrete models. In general, however, the existence of such functions \mathcal{Z}_α is not clear. We settle this problem for all $\kappa \in (0, 4]$ in Theorem 1.1.

To state our result, we have to introduce some definitions. A multiple SLE *partition function* is a positive smooth function $\mathcal{Z}: \mathfrak{X}_{2N} \rightarrow \mathbb{R}_{>0}$ defined on the configuration space

$$\mathfrak{X}_{2N} = \{(x_1, \dots, x_{2N}) \in \mathbb{R}^{2N} \mid x_1 < \dots < x_{2N}\}$$

satisfying the following two properties:

(PDE) *Partial differential equations of second order:*

$$\left[\frac{\kappa}{2} \partial_i^2 + \sum_{j \neq i} \left(\frac{2}{x_j - x_i} \partial_j - \frac{(6 - \kappa)/\kappa}{(x_j - x_i)^2} \right) \right] \mathcal{Z}(x_1, \dots, x_{2N}) = 0 \quad \text{for all } i \in \{1, \dots, 2N\}. \quad (1.1)$$

(COV) *Möbius covariance:* For all Möbius maps φ of \mathbb{H} such that $\varphi(x_1) < \dots < \varphi(x_{2N})$,

$$\mathcal{Z}(x_1, \dots, x_{2N}) = \prod_{i=1}^{2N} \varphi'(x_i)^h \times \mathcal{Z}(\varphi(x_1), \dots, \varphi(x_{2N})), \quad \text{where } h = \frac{6 - \kappa}{2\kappa}. \quad (1.2)$$

Given such a function, one can construct a local $N\text{-SLE}_\kappa$ as discussed in Section 4.3. The above properties (PDE) (1.1) and (COV) (1.2) guarantee that this local multiple SLE process is conformally invariant, the marginal law of one curve with respect to the joint law of all of the curves is a suitably weighted chordal SLE_κ , and that the curves enjoy a certain “commutation”, or “stochastic reparameterization invariance” property — see Section 4.3 and [Dub07, Gra07, KP16] for details.

The *pure partition functions* $\mathcal{Z}_\alpha: \mathfrak{X}_{2N} \rightarrow \mathbb{R}_{>0}$ are indexed by the link patterns $\alpha \in \text{LP}_N$. They are positive solutions to (PDE) (1.1) and (COV) (1.2) that are singled out by boundary conditions given in terms of their asymptotic behavior, determined by the link pattern α :

(ASY) *Asymptotics:* For all $\alpha \in \text{LP}_N$ and for all $j \in \{1, \dots, 2N - 1\}$ and $\xi \in (x_{j-1}, x_{j+2})$,

$$\lim_{x_j, x_{j+1} \rightarrow \xi} \frac{\mathcal{Z}_\alpha(x_1, \dots, x_{2N})}{(x_{j+1} - x_j)^{-2h}} = \begin{cases} 0 & \text{if } \{j, j+1\} \notin \alpha \\ \mathcal{Z}_{\hat{\alpha}}(x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N}) & \text{if } \{j, j+1\} \in \alpha \end{cases} \quad (1.3)$$

where $h = (6 - \kappa)/2\kappa$ and $\hat{\alpha} = \alpha / \{j, j+1\} \in \text{LP}_{N-1}$ denotes the link pattern obtained from α by removing the link $\{j, j+1\}$ and relabeling the remaining indices by $1, 2, \dots, 2N - 2$ (see Figure 1.2).

Attempts to find and classify these functions using Coulomb gas techniques have been made e.g. in [BBK05, Dub06, Dub07, FK15d, KP16]; see also [DF85, FSK15, FSKZ17, LV17]. The main difficulty in the Coulomb gas approach is to show that the constructed functions are positive (whereas smoothness is immediate). On the other hand, as we will see in Section 3, positivity is manifest from the global construction of multiple SLEs, but in this approach, the main obstacle is establishing smoothness. In this article, we combine the approach of [KL07, Law09a] (global construction) with that of [Dub07, Dub15a, Dub15b, KP16] (PDE approach), to establish the existence and uniqueness of local multiple SLEs and their pure partition functions for all $\kappa \in (0, 4]$. This is our first main result.

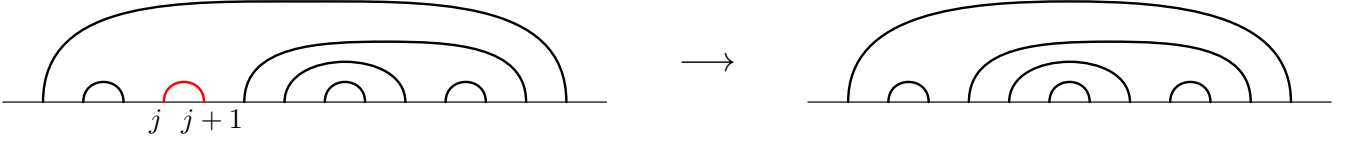


Figure 1.2: The removal of a link from a link pattern (here $j = 4$ and $N = 7$). The left figure is the link pattern $\alpha = \{\{1, 14\}, \{2, 3\}, \{4, 5\}, \{6, 13\}, \{7, 10\}, \{8, 9\}, \{11, 12\}\} \in \text{LP}_7$ and the right figure the link pattern $\alpha/\{4, 5\} = \{\{1, 12\}, \{2, 3\}, \{4, 11\}, \{5, 8\}, \{6, 7\}, \{9, 10\}\} \in \text{LP}_6$.

Theorem 1.1. *Let $\kappa \in (0, 4]$. There exists a unique collection $\{\mathcal{Z}_\alpha: \alpha \in \text{LP}\}$ of smooth functions $\mathcal{Z}_\alpha: \mathfrak{X}_{2N} \rightarrow \mathbb{R}_{>0}$, for $\alpha \in \text{LP}_N$, satisfying the normalization $\mathcal{Z}_\emptyset = 1$ and properties (PDE) (1.1), (COV) (1.2), (ASY) (1.3), and, for all $\alpha = \{\{a_1, b_1\}, \dots, \{a_N, b_N\}\} \in \text{LP}_N$, the power law bound*

$$0 < \mathcal{Z}_\alpha(x_1, \dots, x_{2N}) \leq \prod_{j=1}^N |x_{b_j} - x_{a_j}|^{-2h}, \quad \text{where } h = \frac{6 - \kappa}{2\kappa}. \quad (1.4)$$

The bound (1.4) is very strong. Together with smoothness, the positivity in (1.4) enables us to construct local multiple SLEs, see Corollary 1.2. The upper bound in (1.4) is crucial in our proof of Theorem 1.4 stated below, concerning the connection probabilities of the level lines of the GFF.

For $\kappa = 2$, the existence of the functions \mathcal{Z}_α was known before [KL05, KKP17]. In this case, the positivity and smoothness can be established by identifying \mathcal{Z}_α as scaling limits of connection probabilities for Loop-Erased Random Walks.

It follows from Theorem 1.1 that the functions \mathcal{Z}_α constructed in the previous works [FK15a, KP16] are indeed positive, as conjectured, and agree with the functions of Theorem 1.1 of the present article.

Finally, we point out that above, the pure partition functions \mathcal{Z}_α are only defined for the upper half-plane \mathbb{H} . In other simply connected domains Ω , we extend the definition by conformal covariance:

$$\mathcal{Z}_\alpha(\Omega; \varphi(x_1), \dots, \varphi(x_{2N})) := \prod_{i=1}^{2N} \varphi'(x_i)^{-h} \times \mathcal{Z}_\alpha(x_1, \dots, x_{2N}), \quad (1.5)$$

where $\varphi: \mathbb{H} \rightarrow \Omega$ is any conformal map preserving the order of the marked points. In fact, when $\kappa \leq 8/3$, the functions \mathcal{Z}_α have the following monotonicity property (Proposition 3.4): if $\hat{\Omega} \subset \Omega$ is simply connected and $\hat{\Omega}$ and Ω agree in neighborhoods of x_1, \dots, x_{2N} , then we have

$$\mathcal{Z}_\alpha(\Omega; x_1, \dots, x_{2N}) \geq \mathcal{Z}_\alpha(\hat{\Omega}; x_1, \dots, x_{2N}).$$

Both the global and local definitions of multiple SLEs enjoy conformal invariance and a domain Markov property. However, only in the case of one curve these two properties uniquely determine the SLE_κ . With $N \geq 2$, configurations of curves connecting the marked points $x_1, \dots, x_{2N} \in \partial\Omega$ in the simply connected domain Ω have non-trivial conformal moduli, so their probability measures form a convex set of dimension higher than one. The classification of local multiple SLEs is well established: they are in one-to-one correspondence with (normalized) partition functions [Dub07, KP16]. Thus, we get the following characterization of the convex set of the local $N\text{-SLE}_\kappa$ probability measures.

Corollary 1.2. *Let $\kappa \in (0, 4]$. For any $\alpha \in \text{LP}_N$, there exists a local $N\text{-SLE}_\kappa$ with partition function \mathcal{Z}_α . The functions $\{\mathcal{Z}_\alpha: \alpha \in \text{LP}\}$ are linearly independent. For any $N \geq 1$, the convex hull of the local $N\text{-SLE}_\kappa$ corresponding to \mathcal{Z}_α for $\alpha \in \text{LP}_N$ has dimension $C_N - 1$. The C_N local $N\text{-SLE}_\kappa$ probability measures with pure partition functions \mathcal{Z}_α are the extremal points of this convex set.*

1.2 Global Multiple SLEs

To prove Theorem 1.1, we construct the pure partition functions \mathcal{Z}_α from the Radon-Nikodym derivatives of global multiple SLE measures with respect to product measures of independent SLEs. To this end,

in Theorem 1.3, we give a global construction of multiple SLE_κ measures, for any number of curves and for all possible topological connectivities, when $\kappa \in (0, 4]$. The construction is not new as such: it was done by M. Kozdron and G. Lawler [KL07] in the special case of the rainbow link pattern $\underline{\mathbb{Q}}_N$, illustrated in Figure 3.1 (see also [Dub06, Section 3.4]), and for general link patterns, the construction first appeared in [Law09a, Section 2.7]. However, to prove local commutation of the curves, one needs sufficient regularity that was not established in these articles (for this, see [Dub07, Dub15a, Dub15b]).

In the previous works [KL07, Law09a], the global multiple SLEs were defined in terms of Girsanov reweighting of chordal SLEs. We prefer another definition, where only a minimal amount of characterizing properties are given — in subsequent work [BPW17⁺], we will prove that the global multiple SLEs are uniquely determined by the conditional law property defined below.

Suppose $\Omega \subsetneq \mathbb{C}$ is a non-empty simply connected domain. Fix two boundary points $x, y \in \partial\Omega$. Let $X_0(\Omega; x, y)$ be the collection of continuous simple unparameterized curves in Ω connecting x and y such that they only touch the boundary $\partial\Omega$ in $\{x, y\}$. More generally, fix $N \geq 2$ and a sequence of boundary points x_1, \dots, x_{2N} along $\partial\Omega$ in counterclockwise order. Consider disjoint continuous simple curves in Ω such that each of them connects two points among $\{x_1, \dots, x_{2N}\}$. Again, we encode the connectivities in link patterns $\alpha \in \text{LP}_N$, and we let $X_0^\alpha(\Omega; x_1, \dots, x_{2N})$ be the collection of disjoint curves (η_1, \dots, η_N) , where $\eta_j \in X_0(\Omega; x_{a_j}, x_{b_j})$ for each $j \in \{1, \dots, N\}$.

For any link pattern $\alpha \in \text{LP}_N$, we call a probability measure on $(\eta_1, \dots, \eta_N) \in X_0^\alpha(\Omega; x_1, \dots, x_{2N})$ a *global N -SLE $_\kappa$ associated to α* if, for each $j \in \{1, \dots, N\}$, the conditional law of the curve η_j given $\{\eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_N\}$ is the chordal SLE_κ connecting x_{a_j} and x_{b_j} in the component of the domain $\Omega \setminus \{\eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_N\}$ that contains the endpoints x_{a_j} and x_{b_j} of η_j on its boundary (see Figure 3.2 for an illustration). This definition is natural from the point of view of discrete models: it corresponds to the scaling limit of interfaces with alternating boundary conditions, as described in Sections 1.3 and 1.4.

Theorem 1.3. *Fix $\kappa \in (0, 4]$ and boundary points $x_1, \dots, x_{2N} \in \partial\Omega$ on analytic boundary segments in counterclockwise order along $\partial\Omega$. Then, for any $\alpha \in \text{LP}_N$, there exists a global N -SLE $_\kappa$ associated to α . As a probability measure on the initial segments of the curves, this global N -SLE $_\kappa$ is the same as the local N -SLE $_\kappa$ with partition function \mathcal{Z}_α . It has the following further properties:*

- *If $\hat{\Omega} \subset \Omega$ is simply connected and $\hat{\Omega}$ and Ω agree in neighborhoods of x_1, \dots, x_{2N} , then the global N -SLE $_\kappa$ in $\hat{\Omega}$ is absolutely continuous with respect to the one in Ω with explicit Radon-Nikodym derivative given in Proposition 3.4 in Section 3.*
- *The marginal law of one curve under this global N -SLE $_\kappa$ is absolutely continuous with respect to the chordal SLE_κ with explicit Radon-Nikodym derivative given in Proposition 3.5 in Section 3.*

1.3 $\kappa = 4$: Level Lines of the Gaussian Free Field

The 2D Gaussian Free Field (GFF) is a natural 2-dimensional time analogue of Brownian motion. Like Brownian motion, it is conformally invariant and satisfies a certain domain Markov property. It plays an important role in statistical physics: for instance, it is the scaling limit of the height function of the dimer model [Ken08], and has connections with numerous other models. In the physics literature, the GFF is also known as the free bosonic field, a very fundamental and well-understood object. It is a starting point for many constructions in quantum field theory, see e.g. [DS11] and references therein. In a series of works [SS09, SS13, MS16a], the authors studied the level lines and flow lines of the GFF. The level lines are SLE_4 curves and the flow lines SLE_κ curves for $\kappa \geq 0$. In this article, we focus on the case $\kappa = 4$ and study the connection probabilities of the level lines. In Theorems 1.4 and 1.5, we relate these connection probabilities to the pure partition functions of multiple SLE_4 and find explicit formulas for them.

Fix a constant $\lambda = \pi/2$. Suppose h is a GFF in \mathbb{H} with alternating boundary conditions:

$$-\lambda \text{ on } (x_{2j}, x_{2j+1}), \text{ for } j \in \{0, 1, \dots, N\} \quad \text{and} \quad +\lambda \text{ on } (x_{2j+1}, x_{2j+2}), \text{ for } j \in \{0, 1, \dots, N-1\},$$

with the convention that $x_0 = -\infty$ and $x_{2N+1} = \infty$. For $j \in \{1, \dots, N\}$, let η_j be the level line of h starting from x_{2j-1} , considered as an oriented curve. If x_k is the other endpoint of η_j , we say that the level line η_j terminates at x_k . The endpoints of the level lines (η_1, \dots, η_N) form a planar pair partition, which can be encoded in a link pattern $\mathcal{A} = \mathcal{A}(\eta_1, \dots, \eta_N) \in \text{LP}_N$.

Theorem 1.4. *Consider multiple level lines of the GFF on \mathbb{H} with alternating boundary conditions. For any $\alpha \in \text{LP}_N$, the probability $P_\alpha := \mathbb{P}[\mathcal{A} = \alpha]$ is strictly positive. Conditioned on the event $\{\mathcal{A} = \alpha\}$, the collection $(\eta_1, \dots, \eta_N) \in X_0^\alpha(\mathbb{H}; x_1, \dots, x_{2N})$ is the global N -SLE₄ associated to α constructed in Theorem 1.3. The connection probabilities are explicitly given by*

$$P_\alpha = \frac{\mathcal{Z}_\alpha(x_1, \dots, x_{2N})}{\mathcal{Z}_{\text{GFF}}^{(N)}(x_1, \dots, x_{2N})}, \quad \text{for all } \alpha \in \text{LP}_N, \quad \text{where } \mathcal{Z}_{\text{GFF}}^{(N)} := \sum_{\alpha \in \text{LP}_N} \mathcal{Z}_\alpha \quad (1.6)$$

and \mathcal{Z}_α are the functions of Theorem 1.1 with $\kappa = 4$. For $l, k \in \{1, 2, \dots, 2N\}$, where l is odd and k is even, the probability that the level line of the GFF starting from x_l terminates at x_k is given by

$$P^{(l,k)} = \prod_{\substack{1 \leq j \leq 2N, \\ j \neq l, k}} \left(\frac{|x_j - x_l|}{|x_j - x_k|} \right)^{\delta_j}, \quad \text{where } \delta_j = (-1)^j. \quad (1.7)$$

In order to prove Theorem 1.4, we need good control on the asymptotics of the pure partition functions \mathcal{Z}_α of Theorem 1.1 with $\kappa = 4$. Indeed, the strong bound (1.4) enables us to control end values of certain martingales in Section 5. Note that the property (ASY) (1.3) is not sufficient for this purpose.

An explicit, simple formula for the total partition function $\mathcal{Z}_{\text{GFF}}^{(N)}$ is known [Dub06, KW11a, KP16], see Equation (4.10) in Lemma 4.11. In fact, also the functions \mathcal{Z}_α for $\kappa = 4$, and thus the connection probabilities P_α in (1.6), are explicit. Furthermore, these formulas are in fact algebraic, see Section 6.

Theorem 1.5. *Let $\kappa = 4$. Then, the functions $\{\mathcal{Z}_\alpha : \alpha \in \text{LP}\}$ of Theorem 1.1 can be written as*

$$\mathcal{Z}_\alpha(x_1, \dots, x_{2N}) = \sum_{\beta \in \text{LP}_N} \mathcal{M}_{\alpha, \beta}^{-1} \mathcal{U}_\beta(x_1, \dots, x_{2N}), \quad (1.8)$$

where \mathcal{U}_β are explicit functions defined in Equation (6.2) and the coefficients $\mathcal{M}_{\alpha, \beta}^{-1} \in \mathbb{Z}$ are given in Proposition 6.6 in Section 6.

In [KW11a, KW11b], R. Kenyon and D. Wilson derived formulas for connection probabilities in discrete models (for instance, the double-dimer model) and related these to the multichordal SLE connection probabilities for $\kappa = 2, 4$, and 8 ; see in particular [KW11a, Theorem 5.1]. The scaling limit of chordal interfaces in the double-dimer model is believed to be the multiple SLE₄, and in [KW11a, Theorem 5.1], it was argued that the scaling limits of the double-dimer connection probabilities indeed agree with those of the GFF, i.e., the connection probabilities given by $\mathcal{Z}_\alpha / \mathcal{Z}_{\text{GFF}}^{(N)}$ in Theorem 1.4.

The coefficients $\mathcal{M}_{\alpha, \beta}^{-1}$ appearing in Theorem 1.5 are enumerations of certain combinatorial objects known as “cover-inclusive Dyck tilings” (see Section 6.3). They were first introduced and studied in the articles [KW11a, KW11b, SZ12]. In this approach, one views the link patterns $\alpha \in \text{LP}_N$ equivalently as Dyck paths of $2N$ steps, as illustrated in Figure 6.1 and explained in Section 6.3.

1.4 $\kappa = 3$: Crossing Probabilities in the Ising Model

The 2D Ising model is one of the most studied models of an order-disorder phase transition. Conformal invariance of its scaling limit at criticality in the sense of correlation functions was postulated in the seminal article [BPZ84b]. More recently, S. Smirnov constructed discrete holomorphic observables in the critical Ising model [Smi06]. This offered a way to rigorously establish conformally covariant scaling limits

for all correlation functions [CS12, CI13, CHI15], as well as the conformal invariance of the scaling limit of interfaces in the critical Ising model [CDCH⁺14, HK13, Izy16].

We let discrete domains $(\Omega^\delta; x_1^\delta, \dots, x_{2N}^\delta)$ on the square lattice approximate $(\Omega; x_1, \dots, x_{2N})$ as $\delta \rightarrow 0$ and we consider the critical Ising model in Ω^δ with alternating boundary conditions (see Figure 1.1):

$$\ominus \text{ on } (x_{2j}^\delta, x_{2j+1}^\delta), \text{ for } j \in \{0, 1, \dots, N\} \quad \text{and} \quad \oplus \text{ on } (x_{2j+1}^\delta, x_{2j+2}^\delta), \text{ for } j \in \{0, 1, \dots, N-1\},$$

with the convention that $x_{2N+1} = x_1$. Then, interfaces $(\eta_1^\delta, \dots, \eta_N^\delta)$ connect the boundary points $x_1^\delta, \dots, x_{2N}^\delta$, forming a planar connectivity encoded in a link pattern $\mathcal{A}^\delta \in \text{LP}_N$. We are interested on the scaling limit of the crossing probability $\mathbb{P}[\mathcal{A}^\delta = \alpha]$ for $\alpha \in \text{LP}_N$. For $N = 2$, this limit was derived in [Izy15, Equation (4.4)]. We conjecture that in the above setup, the following is true.

Conjecture 1.6. *We have*

$$\lim_{\delta \rightarrow 0} \mathbb{P}[\mathcal{A}^\delta = \alpha] = \frac{\mathcal{Z}_\alpha(\Omega; x_1, \dots, x_{2N})}{\mathcal{Z}_{\text{Ising}}^{(N)}(\Omega; x_1, \dots, x_{2N})}, \quad \text{where } \mathcal{Z}_{\text{Ising}}^{(N)} := \sum_{\alpha \in \text{LP}_N} \mathcal{Z}_\alpha$$

and \mathcal{Z}_α are the functions defined by (1.5) and Theorem 1.1 with $\kappa = 3$.

The total partition function $\mathcal{Z}_{\text{Ising}}^{(N)}$ has an explicit Pfaffian formula [Izy16, KP16], see Equation (4.9) in Lemma 4.10. However, explicit formulas for \mathcal{Z}_α for $\kappa = 3$ are only known in the cases $N = 1, 2$. In contrast to the case of $\kappa = 4$, for $\kappa = 3$ the formulas are in general not algebraic.

We note that the collection of interfaces of the critical 2D Ising model is related to the global N -SLE₃: conditioned on $\{\mathcal{A}^\delta = \alpha\}$, any scaling limit of $(\eta_1^\delta, \dots, \eta_N^\delta)$ gives a global N -SLE₃ associated to α .

Outline. Section 2 contains preliminary material: the definition and properties of the SLE _{κ} and discussion about the multiple SLE partition functions and solutions of (PDE) (1.1) and (COV) (1.2).

The topic of Section 3 is the construction of global multiple SLEs in order to prove Theorem 1.3. We construct global N -SLE _{κ} probability measures for all link patterns α and for all N in Section 3.1 (Proposition 3.3). The next Section 3.2 gives the boundary perturbation property (Proposition 3.4) and the characterization of the marginal law (Proposition 3.5). We finish the proof of Theorem 1.3 with Lemma 4.15 in Section 4.3 by comparing the two definitions for multiple SLEs — the global and the local.

Section 4 focuses on the pure partition functions \mathcal{Z}_α . Theorem 1.1 concerning the existence and uniqueness of \mathcal{Z}_α is proved in Section 4.1. In Section 4.2, we discuss total partition functions and list explicit formulas for them for $\kappa = 2, 3, 4$. In Section 4.3, we prove Corollary 1.2 and explain the growth process construction of local multiple SLEs and its relationship with the global N -SLE _{κ} probability measures.

The last Sections 5 and 6 focus on the case of $\kappa = 4$. We introduce the Gaussian Free Field and its level lines in Sections 5.1 – 5.2. In Sections 5.3 – 5.4, we find the connection probabilities of the level lines. Theorem 1.4 is proved in Section 5.5.

In Section 6, we derive the explicit formulas of Theorem 1.5 for the multiple SLE₄ pure partition functions, using combinatorics and results from [KW11a, KW11b, KKP17]. We also construct functions called conformal blocks for the GFF and show that they generate multiple SLE₄ processes that can be naturally coupled with the GFF with appropriate boundary conditions, see Proposition 6.12 in Section 6.5.

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2 Preliminaries

This section contains definitions and results from the literature that are needed to understand and prove the main results of this article. In Sections 2.1 and 2.2, we define the chordal SLE _{κ} and give a boundary

perturbation property for it, using a conformally invariant measure known as the Brownian loop measure. Then, in Section 2.3, we discuss the solution space of the system (PDE) (1.1) of second order partial differential equations. We give examples of solutions: multiple SLE partition functions. In Theorem 2.3, we state a result of S. Flores and P. Kleban [FK15b] concerning the asymptotics of solutions, which we use in Section 3 to prove the uniqueness of the pure partition functions of Theorem 1.1. In Proposition 2.5, we prove that all solutions of (PDE) (1.1) are smooth, by showing that this PDE system is hypoelliptic — we follow the idea of J. Dubédat [Dub15a], using the powerful theory of Hörmander [Hör67].

2.1 Schramm-Loewner Evolutions

We call a compact subset K of $\overline{\mathbb{H}}$ an \mathbb{H} -*hull* if $\mathbb{H} \setminus K$ is simply connected. Riemann's Mapping Theorem asserts that there exists a unique conformal map g_K from $\mathbb{H} \setminus K$ onto \mathbb{H} with the property that $\lim_{z \rightarrow \infty} |g_K(z) - z| = 0$. We call such g_K the conformal map from $\mathbb{H} \setminus K$ onto \mathbb{H} *normalized at ∞* .

A *Loewner chain* is a collection of \mathbb{H} -hulls $(K_t, t \geq 0)$ associated with the family of conformal maps $(g_t, t \geq 0)$ obtained by solving the Loewner equation: for each $z \in \mathbb{H}$,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z,$$

where $(W_t, t \geq 0)$ is a real-valued continuous function, which we call the driving function. Let T_z be the *swallowing time* of z defined as $\sup\{t \geq 0 : \inf_{s \in [0, t]} |g_s(z) - W_s| > 0\}$. Denote $K_t := \overline{\{z \in \mathbb{H} : T_z \leq t\}}$. Then g_t is the unique conformal map from $H_t := \mathbb{H} \setminus K_t$ onto \mathbb{H} normalized at ∞ .

Fix $\kappa \geq 0$. The (chordal) *Schramm-Loewner Evolution* SLE_κ in \mathbb{H} from 0 to ∞ is the random Loewner chain $(K_t, t \geq 0)$ driven by $W_t = \sqrt{\kappa} B_t$ where $(B_t, t \geq 0)$ is a standard Brownian motion. S. Rohde and O. Schramm proved in [RS05] that $(K_t, t \geq 0)$ is almost surely generated by a continuous transient curve, i.e., there almost surely exists a continuous curve η such that for each $t \geq 0$, H_t is the unbounded component of $\mathbb{H} \setminus \eta[0, t]$ and $\lim_{t \rightarrow \infty} |\eta(t)| = \infty$. This random curve is the SLE_κ trace in \mathbb{H} from 0 to ∞ . It exhibits phase transitions at $\kappa = 4$ and 8 : the SLE_κ curves are simple when $\kappa \in [0, 4]$ and they have self-touchings when $\kappa > 4$, being space-filling when $\kappa \geq 8$. In this article, we focus on the range $\kappa \in (0, 4]$ when the curve is simple. Its law is a probability measure $\mathbb{P}(\mathbb{H}; 0, \infty)$ on the set $X_0(\mathbb{H}; 0, \infty)$.

The SLE_κ is *conformally invariant*: it is defined in any simply connected domain by pushforward of a conformal map as follows. Given a simply connected domain Ω with two distinct boundary points $x, y \in \partial\Omega$ and any conformal map $\varphi: \mathbb{H} \rightarrow \Omega$ such that $\varphi(0) = x$ and $\varphi(\infty) = y$, we have $\varphi(\eta) \sim \mathbb{P}(\Omega; x, y)$ if $\eta \sim \mathbb{P}(\mathbb{H}; 0, \infty)$, where $\mathbb{P}(\Omega; x, y)$ denotes the law of the SLE_κ in Ω from x to y .

Schramm's classification [Sch00] shows that $\mathbb{P}(\Omega; x, y)$ is the unique probability measure on curves $\eta \in X_0(\Omega; x, y)$ satisfying conformal invariance and the *domain Markov property*: for a stopping time τ , given an initial segment $\eta[0, \tau]$ of the SLE_κ curve $\eta \sim \mathbb{P}(\Omega; x, y)$, the conditional law of the remaining piece $\eta[\tau, \infty)$ is the law $\mathbb{P}(\Omega \setminus \eta[0, \tau]; \eta(\tau), y)$ of the SLE_κ in the remaining domain from the tip $\eta(\tau)$ to y .

We will also use the following *reversibility* of the SLE_κ (for $\kappa \leq 4$) [Zha08a]: the time reversal of the SLE_κ curve $\eta \sim \mathbb{P}(\Omega; x, y)$ in Ω from x to y has the same law $\mathbb{P}(\Omega; y, x)$ as the SLE_κ in Ω from y to x .

Finally, the following change of target point of the chordal SLE_κ will be used in Section 3.

Lemma 2.1. [SW05]. *Fix $\kappa > 0$ and $y > 0$, and let $h = (6 - \kappa)/2\kappa$. Up to the first swallowing time of y , the chordal SLE_κ in \mathbb{H} from 0 to y has the same law as the chordal SLE_κ in \mathbb{H} from 0 to ∞ weighted by the local martingale $N_t = g'_t(y)^h (g_t(y) - W_t)^{-2h}$.*

2.2 Boundary Perturbation of SLE

In the next Lemma 2.2, we recall the boundary perturbation property of the SLE_κ from [LSW03, Section 5]. It gives the Radon-Nikodym derivative between the law of the SLE_κ curve in two simply connected domains $\hat{\Omega} \subset \Omega$ in terms of the Brownian loop measure and the boundary Poisson kernel.

The *Brownian loop measure* is a conformally invariant measure on unrooted Brownian loops in the plane. In the present article, we do not need the precise definition of this measure, so we content ourselves with referring to the literature for the definition: see, e.g., [LW04, Sections 3 and 4] or [FL13]. Given a non-empty simply connected domain $\Omega \subsetneq \mathbb{C}$ and two disjoint subsets $V_1, V_2 \subset \Omega$, we denote by $\mu(\Omega; V_1, V_2)$ the Brownian loop measure of loops in Ω that intersect both V_1 and V_2 . This quantity is a conformal invariant: $\mu(\varphi(\Omega); \varphi(V_1), \varphi(V_2)) = \mu(\Omega; V_1, V_2)$ for any conformal transformation $\varphi: \Omega \rightarrow f(\Omega)$.

In general, the Brownian loop measure is an infinite measure. However, we have $0 \leq \mu(\Omega; V_1, V_2) < \infty$ when both of V_1, V_2 are closed, one of them is compact, $\text{dist}(V_1, V_2) > 0$, and the boundary of the domain Ω is non-polar for the Brownian motion (i.e., there is a positive chance that the Brownian motion started from any point $z \in \mathbb{C}$ hits the set $\partial\Omega$). More generally, for n disjoint subsets V_1, \dots, V_n of Ω , we denote by $\mu(\Omega; V_1, \dots, V_n)$ the Brownian loop measure of loops in Ω that intersect all of V_1, \dots, V_n . Provided that V_j are closed and at least one of them is compact, $\mu(\Omega; V_1, \dots, V_n)$ is finite.

Suppose $x, y \in \partial\Omega$ are two boundary points on analytic segments of $\partial\Omega$. The *boundary Poisson kernel* $H_\Omega(x, y)$ is uniquely characterized by the following two properties (2.1) and (2.2). First, it is conformally covariant: for any conformal map $\varphi: \Omega \rightarrow \varphi(\Omega)$, we have

$$\varphi'(x)\varphi'(y)H_{\varphi(\Omega)}(\varphi(x), \varphi(y)) = H_\Omega(x, y). \quad (2.1)$$

Second, for the upper-half plane with $x, y \in \mathbb{R}$, we have the explicit formula (we do not include π^{-1} here)

$$H_{\mathbb{H}}(x, y) = |y - x|^{-2}. \quad (2.2)$$

In addition, for two simply connected domains $\hat{\Omega} \subset \Omega$ that agree in neighborhoods of x and y , we have

$$H_{\hat{\Omega}}(x, y) \leq H_\Omega(x, y). \quad (2.3)$$

Lemma 2.2. *Fix $\kappa \in (0, 4]$ and let $h = (6 - \kappa)/2\kappa$ and $c = (3\kappa - 8)(6 - \kappa)/2\kappa$. Let Ω be a simply connected domain with non-polar boundary, and let $x, y \in \partial\Omega$ be distinct boundary points on analytic boundary segments. Assume that $\hat{\Omega} \subset \Omega$ is simply connected and that $\hat{\Omega}$ and Ω agree in neighborhoods of x and y . Then, the SLE $_\kappa$ in $\hat{\Omega}$ from x to y is absolutely continuous with respect to the SLE $_\kappa$ in Ω from x to y , with Radon-Nikodym derivative given by*

$$\frac{d\mathbb{P}(\hat{\Omega}; x, y)}{d\mathbb{P}(\Omega; x, y)} = \left(\frac{H_\Omega(x, y)}{H_{\hat{\Omega}}(x, y)} \right)^h \mathbb{1}_{\{\eta \subset \hat{\Omega}\}} \exp(c\mu(\Omega; \eta, \Omega \setminus \hat{\Omega})).$$

Proof. See [LSW03, Section 5] and [KL07, Proposition 3.1]. □

2.3 Solutions to the Second Order PDE System (PDE)

In this section, we present known facts about the solution space of the system (PDE) (1.1) of partial differential equations of second order. Particular examples of solutions are the multiple SLE partition functions, and we give examples of known formulas for them. We also state a crucial result from [FK15b] concerning the asymptotics of solutions. This result, Theorem 2.3, says that solutions to (PDE) (1.1) and (COV) (1.2) having certain asymptotic properties must vanish. We use this property in Section 4 to prove the uniqueness of the pure partition functions. Finally, we discuss regularity of the solutions to the system (PDE) (1.1). In Proposition 2.5, we prove that these PDEs are hypoelliptic, that is, all distributional solutions for them are in fact smooth functions. This result was proved in [Dub15a] using the powerful theory of Hörmander [Hör67], which we also briefly recall.

2.3.1 Examples of Partition Functions

Fix $\kappa \in (0, 8)$ and $h = (6 - \kappa)/2\kappa$. The pure partition functions for $N = 1$ and $N = 2$ can be found by a calculation. The case $N = 1$ is almost trivial: then we have, for $x < y$ and $\frown = \{\{1, 2\}\}$,

$$\mathcal{Z}^{(1)}(x, y) = \mathcal{Z}_{\frown}(x, y) = (y - x)^{-2h}.$$

When $N = 2$, the system (PDE) (1.1) with the Möbius covariance (COV) (1.2) reduces to an ordinary differential equation (ODE), since we can fix three out of the four degrees of freedom. This ODE is the hypergeometric equation, whose solutions are well-known. With the boundary conditions (ASY) (1.3), we obtain for $\overbrace{\quad\quad\quad} = \{\{1, 4\}, \{2, 3\}\}$ and $\underbrace{\quad\quad\quad} = \{\{1, 2\}, \{3, 4\}\}$, and for $x_1 < x_2 < x_3 < x_4$,

$$\begin{aligned}\mathcal{Z}_{\overbrace{\quad\quad\quad}}(x_1, x_2, x_3, x_4) &= (x_4 - x_1)^{-2h}(x_3 - x_2)^{-2h}q^{2/\kappa}F(q), \\ \mathcal{Z}_{\underbrace{\quad\quad\quad}}(x_1, x_2, x_3, x_4) &= (x_2 - x_1)^{-2h}(x_4 - x_3)^{-2h}(1 - q)^{2/\kappa}F(1 - q),\end{aligned}$$

where q is a cross-ratio and F is a hypergeometric function:

$$q = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_4 - x_2)(x_3 - x_1)}, \quad F(z) := {}_2F_1\left(\frac{4}{\kappa}, 1 - \frac{4}{\kappa}, \frac{8}{\kappa}; z\right).$$

Note that F is bounded on $[0, 1]$ when $\kappa \in (0, 8)$. For some parameter values, these formulas are algebraic:

$$\text{For } \kappa = 2, \quad \mathcal{Z}_{\overbrace{\quad\quad\quad}}(x_1, x_2, x_3, x_4) = (x_4 - x_1)^{-2}(x_3 - x_2)^{-2}q(2 - q). \quad (2.4)$$

$$\text{For } \kappa = 4, \quad \mathcal{Z}_{\overbrace{\quad\quad\quad}}(x_1, x_2, x_3, x_4) = (x_4 - x_1)^{-1/2}(x_3 - x_2)^{-1/2}q^{1/2}. \quad (2.5)$$

$$\text{For } \kappa = 16/3, \quad \mathcal{Z}_{\overbrace{\quad\quad\quad}}(x_1, x_2, x_3, x_4) = (x_4 - x_1)^{-1/4}(x_3 - x_2)^{-1/4}q^{3/8}(1 + \sqrt{1 - q})^{-1/2}. \quad (2.6)$$

When $\kappa = 4$, Equation (2.5) gives

$$\frac{\mathcal{Z}_{\overbrace{\quad\quad\quad}}(x_1, x_2, x_3, x_4)}{\mathcal{Z}_{\overbrace{\quad\quad\quad}}(x_1, x_2, x_3, x_4) + \mathcal{Z}_{\underbrace{\quad\quad\quad}}(x_1, x_2, x_3, x_4)} = q.$$

The right hand side is the same as the connection probability of level lines in GFF, see Lemma 5.3.

When $\kappa = 16/3$, Equation (2.6) gives

$$\frac{\mathcal{Z}_{\overbrace{\quad\quad\quad}}(x_1, x_2, x_3, x_4)}{\mathcal{Z}_{\overbrace{\quad\quad\quad}}(x_1, x_2, x_3, x_4) + \mathcal{Z}_{\underbrace{\quad\quad\quad}}(x_1, x_2, x_3, x_4)} = \frac{\sqrt{1 - \sqrt{1 - q}}}{\sqrt{1 - \sqrt{1 - q}} + \sqrt{1 - \sqrt{q}}}.$$

The right hand side is the critical FK-Ising crossing probability derived in [CS12, Theorem C].

2.3.2 A Crucial Uniqueness Result

The following theorem is a deep result due to S. Flores and P. Kleban. It is formulated as a lemma in the series [FK15a, FK15b, FK15c, FK15d] of articles, which concerns the dimension of the solution space of (PDE) (1.1) and (COV) (1.2) under a condition (2.7) of power law growth given below. The proof of this lemma constitutes the whole article [FK15b], relying on the theory of elliptic partial differential equations, Green function techniques, and careful estimates on the asymptotics of the solutions.

Theorem 2.3. [FK15b, Lemma 1]. *Fix $\kappa \in (0, 8)$. Let $F: \mathfrak{X}_{2N} \rightarrow \mathbb{C}$ be a function satisfying properties (PDE) (1.1) and (COV) (1.2). Suppose furthermore that there exist constants $C > 0$ and $p > 0$ such that for all $N \geq 1$ and $(x_1, \dots, x_{2N}) \in \mathfrak{X}_{2N}$, we have*

$$|F(x_1, \dots, x_{2N})| \leq C \prod_{1 \leq i < j \leq 2N} (x_j - x_i)^{\mu_{ij}(p)}, \quad \text{where } \mu_{ij}(p) := \begin{cases} p & \text{if } |x_j - x_i| > 1, \\ -p & \text{if } |x_j - x_i| < 1. \end{cases} \quad (2.7)$$

If F also has the asymptotics property

$$\lim_{x_j, x_{j+1} \rightarrow \xi} \frac{F(x_1, \dots, x_{2N})}{(x_{j+1} - x_j)^{-2h}} = 0 \quad \text{for all } j \in \{2, \dots, 2N - 1\} \text{ and } \xi \in (x_{j-1}, x_{j+2})$$

(with the convention that $x_0 = -\infty$ and $x_{2N+1} = +\infty$), then $F \equiv 0$.

Motivated by Theorem 2.3, we define the following solution space of the system (PDE) (1.1):

$$\mathcal{S}_N := \{F: \mathfrak{X}_{2N} \rightarrow \mathbb{C} \mid F \text{ satisfies (PDE) (1.1), (COV) (1.2), and (2.7)}\}. \quad (2.8)$$

We use this notation throughout. The bound (2.7) is easy to verify for the solutions studied in the present article. Hence, Theorem 2.3 gives us the uniqueness of the pure partition functions for Theorem 1.1.

2.3.3 Hypoellipticity

Following [Dub15a, Lemma 5], we prove next that any distributional solution to the system (PDE) (1.1) is necessarily smooth. This follows from the fact that the PDE system (1.1) is hypoelliptic, for it satisfies the Hörmander bracket condition. For details concerning hypoelliptic PDEs, see e.g. [Str08, Chapter 7].

For an open set $O \subset \mathbb{R}^n$ and a field \mathbb{F} (which in our case is either \mathbb{R} or \mathbb{C}), we denote by $C^\infty(O; \mathbb{F})$ the set of smooth functions from O to \mathbb{F} . We also denote by $S(O; \mathbb{F})$ the usual Schwartz space of rapidly decreasing functions from O to \mathbb{F} , and by $S'(O; \mathbb{F})$ the space of tempered distributions, that is, the dual space of $S(O; \mathbb{F})$. Let \mathcal{D} be a linear partial differential operator with real analytic coefficients defined on an open set $U \subset \mathbb{R}^n$. The operator \mathcal{D} is said to be *hypoelliptic* on U if for every open set $O \subset U$, the following holds: if $F \in S'(O; \mathbb{C})$ satisfies $\mathcal{D}F \in C^\infty(O; \mathbb{C})$, then we have $F \in C^\infty(O; \mathbb{C})$.

Given a linear partial differential operator, how to prove that it is hypoelliptic? For operators of certain form, L. Hörmander proved in [Hör67] a powerful characterization for hypoellipticity. Suppose $U \subset \mathbb{R}^n$ is an open set, denote $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, and consider smooth vector fields

$$X_j := \sum_{k=1}^n a_{jk}(\mathbf{x}) \partial_k, \quad \text{for } j \in \{0, 1, \dots, m\}, \quad (2.9)$$

where $a_{jk} \in C^\infty(U; \mathbb{R})$ are smooth real-valued coefficients. Hörmander's theorem gives a characterization for hypoellipticity of partial differential operators of the form

$$\mathcal{D} = \sum_{j=1}^m X_j^2 + X_0 + c(\mathbf{x}), \quad (2.10)$$

where $c \in C^\infty(U; \mathbb{R})$. Denote by \mathfrak{g} the real Lie algebra generated by the vector fields (2.9), and for $\mathbf{x} \in U$, let $\mathfrak{g}_{\mathbf{x}} \subset T_{\mathbf{x}}\mathbb{R}^n$ be the subspace of the tangent space of \mathbb{R}^n obtained by evaluating the elements of \mathfrak{g} at \mathbf{x} .

Theorem 2.4. [Hör67, Theorem 1.1]. *Let $U \subset \mathbb{R}^n$ be an open set and X_0, \dots, X_m vector fields as in (2.9). If for all $\mathbf{x} \in U$, the rank of $\mathfrak{g}_{\mathbf{x}}$ equals n , then the operator \mathcal{D} of the form (2.10) is hypoelliptic on U .*

Consider now the partial differential operators appearing in the system (PDE) (1.1). They are defined on the open set $\mathfrak{U}_{2N} = \{(x_1, \dots, x_{2N}) \in \mathbb{R}^{2N} \mid x_i \neq x_j \text{ for all } i \neq j\}$. The following result was proved in [Dub15a, Lemma 5] in a very general setup. For clarity, we give the proof in our simple case.

Proposition 2.5. *The operator $\mathcal{D}^{(i)} = \frac{\kappa}{2} \partial_i^2 + \sum_{j \neq i} \left(\frac{2}{x_j - x_i} \partial_j - \frac{(6-\kappa)/\kappa}{(x_j - x_i)^2} \right)$, for $i \in \{1, \dots, 2N\}$, is hypoelliptic. In particular, any distributional solution $F \in S'(\mathfrak{U}_{2N}; \mathbb{C})$ to $\mathcal{D}^{(i)}F = 0$ is smooth: $F \in C^\infty(\mathfrak{U}_{2N}; \mathbb{C})$.*

Proof. Observe that choosing $X_0 = \sum_{j \neq i} \frac{2}{x_j - x_i} \partial_j$, $X_1 = \sqrt{\frac{\kappa}{2}} \partial_i$, and $c(\mathbf{x}) = \sum_{j \neq i} \frac{(\kappa-6)/\kappa}{(x_j - x_i)^2}$, the operator $\mathcal{D}^{(i)}$ is of the form (2.10). Thus, by Theorem 2.4, we only need to check that at any $\mathbf{x} = (x_1, \dots, x_{2N}) \in \mathfrak{U}_{2N}$, the vector fields X_0 and X_1 and their commutators at \mathbf{x} generate a vector space of dimension $2N$. For this, without loss of generality, we let $i = 1$, and we define the k -fold commutator of X_0 with itself by $X_0^{[0]} := X_0$ for $k = 0$ and $X_0^{[k]} := [X_0, X_0^{[k-1]}]$ for $k \geq 1$. Now, for all $k \geq 0$, we have

$$X_0^{[k]} = \sum_{j \neq i} \left(\frac{2}{x_j - x_1} \right)^{k+1},$$

so we can write $(X_0^{[0]}, \dots, X_0^{[2N-2]})^t = A(\partial_2, \dots, \partial_{2N})^t$, where $A = (A_{kl})$ with $A_{kl} = (2(x_l - x_1)^{-1})^k$ for $k, l \in \{1, \dots, 2N-1\}$ is a Vandermonde type matrix, whose determinant is non-zero. Thus, we have $\partial_1 = \sqrt{\frac{2}{\kappa}} X_1$ and we can solve for $\partial_2, \dots, \partial_{2N}$ in terms of $X_0^{[0]}, \dots, X_0^{[2N-2]}$. This concludes the proof. \square

2.3.4 Dual Elements

To finish this section, we consider certain linear functionals $\mathcal{L}_\alpha : \mathcal{S}_N \rightarrow \mathbb{C}$ on the solution space \mathcal{S}_N defined in (2.8). It was proved in the series [FK15a, FK15b, FK15c, FK15d] of articles that $\dim(\mathcal{S}_N) = C_N$. The linear functionals \mathcal{L}_α were defined in [FK15a], where they were called allowable sequences of limits (see also [KP16]). In fact, for each N , they form a dual basis for the multiple SLE pure partition functions $\{\mathcal{Z}_\alpha : \alpha \in \text{LP}_N\}$ — see Proposition 4.6. To define these linear functionals, we consider a link pattern

$$\alpha = \{\{a_1, b_1\}, \dots, \{a_N, b_N\}\} \in \text{LP}_N$$

with its link ordered as $\{a_1, b_1\}, \dots, \{a_N, b_N\}$, where we take $a_j < b_j$ for all $j \in \{1, \dots, N\}$ by convention. We consider successive removals of links of the form $\{j, j+1\}$ from α . Recall that the link pattern obtained from α by removing the link $\{j, j+1\}$ is denoted by $\alpha/\{j, j+1\}$, as illustrated in Figure 1.2. Note that after the removal, the indices of the remaining links have to be relabeled by $1, 2, \dots, 2N-2$. The ordering of links in α is said to be *allowable* if all links of α can be removed in the order $\{a_1, b_1\}, \dots, \{a_N, b_N\}$ in such a way that at each step, the link to be removed connects two consecutive indices, as illustrated in Figure 2.1 (see e.g. [KP16, Section 3.5] for a more formal definition).

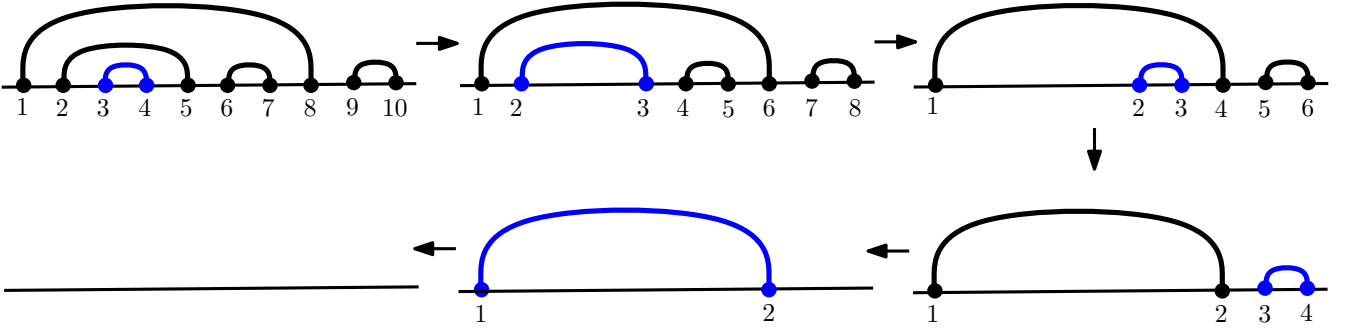


Figure 2.1: An allowable ordering of links in a link pattern α and the corresponding link removals.

Suppose the ordering $\{a_1, b_1\}, \dots, \{a_N, b_N\}$ of the links of α is allowable. Fix points $\xi_j \in (x_{a_j-1}, x_{b_j+1})$ for all $j \in \{1, \dots, N\}$, with the convention that $x_0 = -\infty$ and $x_{2N+1} = +\infty$. It was proved in [FK15a, Section III] that the following sequence of limits exists for any solution $F \in \mathcal{S}_N$:

$$\mathcal{L}_\alpha(F) := \lim_{x_{a_N}, x_{b_N} \rightarrow \xi_N} \cdots \lim_{x_{a_1}, x_{b_1} \rightarrow \xi_1} (x_{b_N} - x_{a_N})^{2h} \cdots (x_{b_1} - x_{a_1})^{2h} F(x_1, \dots, x_{2N}). \quad (2.11)$$

Furthermore, any other allowable ordering of the links of α gives the same limit (2.11). Therefore, for each $\alpha \in \text{LP}_N$ with any choice of allowable ordering of links, Equation (2.11) defines a linear functional $\mathcal{L}_\alpha : \mathcal{S}_N \rightarrow \mathbb{C}$.

3 Global Multiple SLEs

Fix $\kappa \in (0, 4]$. Let Ω be a simply connected domain with $2N$ distinct boundary points x_1, \dots, x_{2N} appearing in counterclockwise order along $\partial\Omega$. Suppose that all $x_j \in \partial\Omega$ lie on analytic boundary segments of Ω . In this section, we construct probability measures $\mathbb{Q}_\alpha^\#$ on the set $X_0^\alpha(\Omega; x_1, \dots, x_{2N})$ of disjoint, continuous simple curves (η_1, \dots, η_N) in Ω such that, given a link pattern $\alpha = \{\{a_1, b_1\}, \dots, \{a_N, b_N\}\}$, for each $j \in \{1, \dots, N\}$, the curve η_j connects x_{a_j} to x_{b_j} according to α .

In [KL07] M. Kozdron and G. Lawler constructed such a probability measure in the special case when the curves form the rainbow connectivity, illustrated in Figure 3.1, encoded in the link pattern $\underline{\alpha}_N = \{\{1, 2N\}, \{2, 2N-1\}, \dots, \{N, N+1\}\}$ (see also [Dub06, Section 3.4]). The generalization of this construction to the case of any possible topological connectivity of the curves, encoded in a general link pattern $\alpha \in \text{LP}_N$, was stated in Lawler's works [Law09a, Law09b], but without proof.

In the present article, we give a combinatorial construction of these probability measures on curves, which appears to agree with [Law09a, Section 2.7]. In contrast to the previous works, we formulate the result focusing on the conceptual definition of the global multiple SLEs instead of just defining them as weighted chordal SLEs. These global N -SLE $_{\kappa}$ measures have the defining property that for each $j \in \{1, \dots, N\}$, the conditional law of η_j given $\{\eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_N\}$ is the chordal SLE $_{\kappa}$ connecting x_{a_j} and x_{b_j} in the connected component $\hat{\Omega}_j$ of the domain $\Omega \setminus \{\eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_N\}$ having x_{a_j} and x_{b_j} on its boundary. In subsequent work [BPW17⁺], we will prove that this property uniquely determines the global multiple SLE measures. The uniqueness of the global 2-SLE $_{\kappa}$ for $\kappa \in (0, 4]$ was proved in [MS16b, Theorem 4.1]. In the special case $\kappa = 4$, the uniqueness of the global N -SLE $_4$ can be proved by similar arguments, as explained in [Wu17, Remark 4.4].

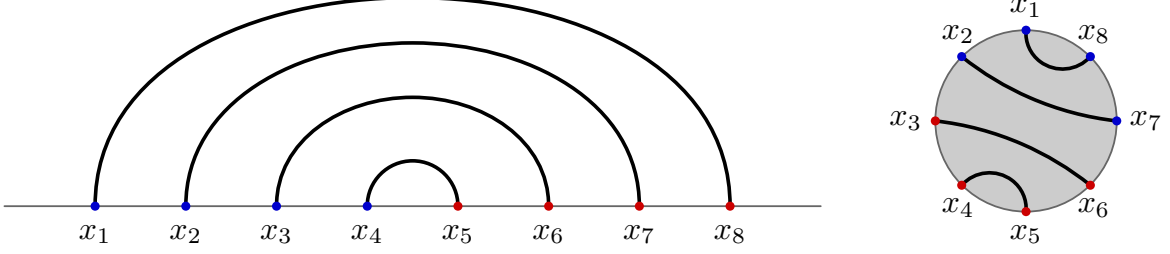


Figure 3.1: The rainbow link pattern with four links, denoted by \underline{n}_4 .

Theorem 1.3. *Fix $\kappa \in (0, 4]$ and boundary points $x_1, \dots, x_{2N} \in \partial\Omega$ on analytic boundary segments in counterclockwise order along $\partial\Omega$. Then, for any $\alpha \in \text{LP}_N$, there exists a global N -SLE $_{\kappa}$ associated to α . As a probability measure on the initial segments of the curves, this global N -SLE $_{\kappa}$ is the same as the local N -SLE $_{\kappa}$ with partition function \mathcal{Z}_{α} . It has the following further properties:*

- *If $\hat{\Omega} \subset \Omega$ is simply connected and $\hat{\Omega}$ and Ω agree in neighborhoods of x_1, \dots, x_{2N} , then the global N -SLE $_{\kappa}$ in $\hat{\Omega}$ is absolutely continuous with respect to the one in Ω with explicit Radon-Nikodym derivative given in Proposition 3.4 in Section 3.*
- *The marginal law of one curve under this global N -SLE $_{\kappa}$ is absolutely continuous with respect to the chordal SLE $_{\kappa}$ with explicit Radon-Nikodym derivative given in Proposition 3.5 in Section 3.*

Proof. We prove the existence of a global N -SLE $_{\kappa}$ associated to α by constructing it in Proposition 3.3 in Section 3.1. The two properties of the measure are proved in Propositions 3.4 and 3.5 in Section 3.2. Finally, in Lemma 4.15 in Section 4.3, we prove that the local and global SLE $_{\kappa}$ associated to α agree. \square

3.1 Construction of Global Multiple SLEs

The general idea to construct global multiple SLEs is that one defines the measure by its Radon-Nikodym derivative with respect to the product measure of independent chordal SLEs. This Radon-Nikodym derivative can be written in terms of the Brownian loop measure. The same idea can also be used to construct multiple SLEs in finitely connected domains, see [Law09a, Law09b, Law11].

Fix $\alpha = \{\{a_1, b_1\}, \dots, \{a_N, b_N\}\} \in \text{LP}_N$. To construct the global N -SLE $_{\kappa}$ associated to α , we introduce a combinatorial expression of Brownian loop measures, denoted by m_{α} . For each configuration $(\eta_1, \dots, \eta_N) \in X_0^{\alpha}(\Omega; x_1, \dots, x_{2N})$, we note that $\Omega \setminus \{\eta_1, \dots, \eta_N\}$ has $N + 1$ connected components (c.c). The boundary of each c.c. \mathcal{C} contains some of the curves $\{\eta_1, \dots, \eta_N\}$. We denote by

$$\mathcal{B}(\mathcal{C}) := \{j \in \{1, \dots, N\} : \eta_j \subset \partial\mathcal{C}\}$$

the set of indices j specified by the curves $\eta_j \subset \partial\mathcal{C}$. If $\mathcal{B}(\mathcal{C}) = \{j_1, \dots, j_p\}$, we define

$$m(\mathcal{C}) := \sum_{\substack{i_1, i_2 \in \mathcal{B}(\mathcal{C}), \\ i_1 \neq i_2}} \mu(\Omega; \eta_{i_1}, \eta_{i_2}) - \sum_{\substack{i_1, i_2, i_3 \in \mathcal{B}(\mathcal{C}), \\ i_1 \neq i_2 \neq i_3 \neq i_1}} \mu(\Omega; \eta_{i_1}, \eta_{i_2}, \eta_{i_3}) + \dots + (-1)^p \mu(\Omega; \eta_{j_1}, \dots, \eta_{j_p}).$$

For $(\eta_1, \dots, \eta_N) \in X_0^\alpha(\Omega; x_1, \dots, x_{2N})$, we define

$$m_\alpha(\Omega; \eta_1, \dots, \eta_N) := \sum_{\text{c.c. } \mathcal{C} \text{ of } \Omega \setminus \{\eta_1, \dots, \eta_N\}} m(\mathcal{C}). \quad (3.1)$$

If α is the rainbow pattern \underline{m}_N , then the quantity m_α has a simple expression:

$$m_{\underline{m}_N}(\Omega; \eta_1, \dots, \eta_N) = \sum_{j=1}^{N-1} \mu(\Omega; \eta_j, \eta_{j+1}), \quad \text{for } \underline{m}_N = \{\{1, 2N\}, \{2, 2N-1\}, \dots, \{N, N+1\}\}.$$

More generally, m_α is given by an inclusion-exclusion procedure that depends on α . It has the following cascade property, which will be crucial in the sequel.

Lemma 3.1. *Let $\alpha \in \text{LP}_N$ and $j \in \{1, \dots, N\}$, and denote $\hat{\alpha} = \alpha / \{a_j, b_j\} \in \text{LP}_{N-1}$. Then we have*

$$m_\alpha(\Omega; \eta_1, \dots, \eta_N) = m_{\hat{\alpha}}(\Omega; \eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_N) + \mu(\Omega; \eta_j, \Omega \setminus \hat{\Omega}_j),$$

where $\hat{\Omega}_j$ is the connected component of $\Omega \setminus \{\eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_N\}$ having x_{a_j} and x_{b_j} on its boundary.

Proof. As illustrated in Figure 3.2, the domain $\Omega \setminus \{\eta_1, \dots, \eta_N\}$ has $N+1$ connected components, two of which have the curve η_j on their boundary. We denote them by \mathcal{C}_j^L and \mathcal{C}_j^R . We split the summation in m_α into two parts, depending on whether or not η_j is a part of the boundary of the connected component \mathcal{C} :

$$m_\alpha(\Omega; \eta_1, \dots, \eta_N) = S_1 + S_2, \quad \text{where} \quad S_1 = m(\mathcal{C}_j^L) + m(\mathcal{C}_j^R) \quad \text{and} \quad S_2 = \sum_{\mathcal{C} : j \notin \mathcal{B}(\mathcal{C})} m(\mathcal{C}).$$

The quantity S_1 is a sum of terms of the form $\mu(\Omega; \eta_{i_1}, \dots, \eta_{i_k})$. We split the terms in $S_1 = S_{1,1} + S_{1,2}$ into two parts: $S_{1,1}$ is the sum of the terms in S_1 including η_j and $S_{1,2}$ is the sum of the terms in S_1 excluding η_j . Now we have $m_\alpha(\Omega; \eta_1, \dots, \eta_N) = S_{1,1} + S_{1,2} + S_2$.

On the other hand, by definition (3.1), the quantity $m_{\hat{\alpha}}$ can be written in the form

$$m_{\hat{\alpha}}(\Omega; \eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_N) = S_2 + S_{1,2} + S_3,$$

where S_3 contains the contribution of terms of type $\mu(\Omega; \eta_{i_1}, \dots, \eta_{i_k})$ for curves $\eta_{i_1}, \dots, \eta_{i_k}$ such that $i_1, \dots, i_k \in \mathcal{B}(\hat{\Omega}_j)$ and at least two of these curves belong to different $\partial\mathcal{C}_j^L, \partial\mathcal{C}_j^R$. For such curves, any Brownian loop intersecting all of them must also intersect η_j . Thus, we have $\mu(\Omega; \eta_j, \Omega \setminus \hat{\Omega}_j) = S_{1,1} - S_3$. The asserted identity follows:

$$\begin{aligned} m_\alpha(\Omega; \eta_1, \dots, \eta_N) &= S_{1,1} - S_3 + S_3 + S_{1,2} + S_2 \\ &= \mu(\Omega; \eta_j, \Omega \setminus \hat{\Omega}_j) + m_{\hat{\alpha}}(\Omega; \eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_N). \end{aligned}$$

□

Next, we record a boundary perturbation property for the quantity m_α , also needed later.

Lemma 3.2. *Suppose K is a relatively compact subset of Ω such that $\Omega \setminus K$ is simply connected, and assume that the distance between K and $\{\eta_1, \dots, \eta_N\}$ is strictly positive. Then we have*

$$m_\alpha(\Omega; \eta_1, \dots, \eta_N) = m_\alpha(\Omega \setminus K; \eta_1, \dots, \eta_N) + \sum_{j=1}^N \mu(\Omega; K, \eta_j) - \mu\left(\Omega; K, \bigcup_{j=1}^N \eta_j\right). \quad (3.2)$$

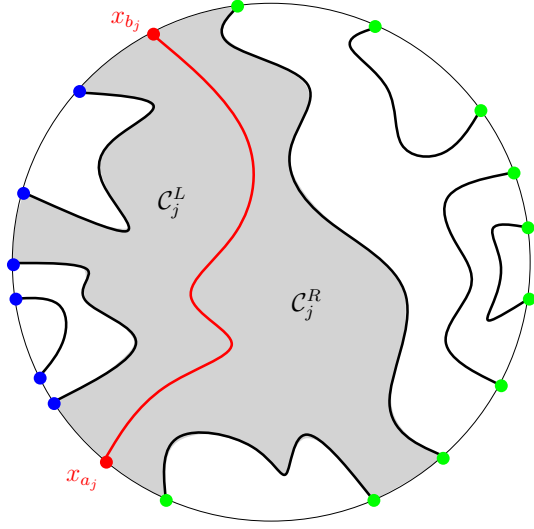


Figure 3.2: Illustration of notations used throughout. The red curve is η_j . The domain $\Omega \setminus \{\eta_1, \dots, \eta_N\}$ has $N + 1$ connected components. Two of them have η_j on their boundary, denoted by \mathcal{C}_j^L and \mathcal{C}_j^R . The grey domain is $\hat{\Omega}_j$, that is, the connected component of $\Omega \setminus \{\eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_N\}$ having the endpoints $x_{a_j}, x_{b_j} \in \partial\hat{\Omega}_j$ of the curve η_j on its boundary. On the other hand, in Proposition 3.5 we denote by D_j^L and D_j^R the two connected components of $\Omega \setminus \eta_j$ lying on the left and right of the curve, respectively. The sub-link patterns of α associated to these two components are denoted by α_j^L and α_j^R , and illustrated respectively in blue and green in the figure.

Proof. We prove the asserted identity by induction on N . For $N = 1$, we have $m_{\hat{\alpha}}(\Omega; \eta) = 0$, so the claim is clear. Assume that (3.2) holds for all link patterns in LP_{N-1} , denote $\hat{\alpha} = \alpha / \{x_{a_1}, x_{b_1}\} \in \text{LP}_{N-1}$, and suppose that η_1 is the curve from x_{a_1} to x_{b_1} . Finally, let $\hat{\Omega}_1$ be the connected component of $\Omega \setminus \{\eta_2, \dots, \eta_N\}$ having the endpoints of η_1 on its boundary (as in Figure 3.2). Using Lemma 3.1 and the obvious fact that $\mu(\Omega; \eta_1, \Omega \setminus \hat{\Omega}_1) = \mu(\Omega \setminus K; \eta_1, \Omega \setminus \hat{\Omega}_1) + \mu(\Omega; K, \eta_1, \Omega \setminus \hat{\Omega}_1)$, we can write m_{α} in the form

$$\begin{aligned} m_{\alpha}(\Omega; \eta_1, \dots, \eta_N) &= m_{\hat{\alpha}}(\Omega; \eta_2, \dots, \eta_N) + \mu(\Omega; \eta_1, \Omega \setminus \hat{\Omega}_1) \\ &= m_{\hat{\alpha}}(\Omega; \eta_2, \dots, \eta_N) + \mu(\Omega \setminus K; \eta_1, \Omega \setminus \hat{\Omega}_1) + \mu(\Omega; K, \eta_1, \Omega \setminus \hat{\Omega}_1). \end{aligned}$$

By the induction hypothesis, for $\hat{\alpha} \in \text{LP}_{N-1}$, we have

$$m_{\hat{\alpha}}(\Omega; \eta_2, \dots, \eta_N) = m_{\hat{\alpha}}(\Omega \setminus K; \eta_2, \dots, \eta_N) + \sum_{j=2}^N \mu(\Omega; K, \eta_j) - \mu\left(\Omega; K, \bigcup_{j=2}^N \eta_j\right).$$

Combining these two relations with Lemma 3.1, we obtain

$$\begin{aligned} m_{\alpha}(\Omega; \eta_1, \dots, \eta_N) &= m_{\alpha}(\Omega \setminus K; \eta_1, \dots, \eta_N) + \sum_{j=2}^N \mu(\Omega; K, \eta_j) - \mu\left(\Omega; K, \bigcup_{j=2}^N \eta_j\right) \\ &\quad + \mu(\Omega; K, \eta_1, \Omega \setminus \hat{\Omega}_1). \end{aligned}$$

Note now that $\mu(\Omega; K, \eta_1, \Omega \setminus \hat{\Omega}_1) = \mu\left(\Omega; K, \eta_1, \bigcup_{j=2}^N \eta_j\right)$, so

$$\begin{aligned} \mu\left(\Omega; K, \bigcup_{j=1}^N \eta_j\right) &= \mu(\Omega; K, \eta_1) + \mu\left(\Omega; K, \bigcup_{j=2}^N \eta_j\right) - \mu\left(\Omega; K, \eta_1, \bigcup_{j=2}^N \eta_j\right) \\ &= \mu(\Omega; K, \eta_1) + \mu\left(\Omega; K, \bigcup_{j=2}^N \eta_j\right) - \mu(\Omega; K, \eta_1, \Omega \setminus \hat{\Omega}_1). \end{aligned}$$

Combining the above two equations, we get the asserted identity (3.2):

$$m_{\alpha}(\Omega; \eta_1, \dots, \eta_N) = m_{\alpha}(\Omega \setminus K; \eta_1, \dots, \eta_N) + \sum_{j=2}^N \mu(\Omega; K, \eta_j) - \mu\left(\Omega; K, \bigcup_{j=1}^N \eta_j\right) + \mu(\Omega; K, \eta_1).$$

□

Now, we are ready to construct the probability measure of Theorem 1.3.

Proposition 3.3. *Fix $\kappa \in (0, 4]$ and boundary points $x_1, \dots, x_{2N} \in \partial\Omega$ on analytic boundary segments in counterclockwise order along $\partial\Omega$. Then, for any $\alpha \in \text{LP}_N$, there exists a global N -SLE $_\kappa$ associated to α .*

Proof. For $\alpha = \{\{a_1, b_1\}, \dots, \{a_N, b_N\}\} \in \text{LP}_N$, let \mathbb{P}_α denote the product measure

$$\mathbb{P}_\alpha := \bigotimes_{j=1}^N \mathbb{P}(\Omega; x_{a_j}, x_{b_j})$$

of N independent chordal SLE $_\kappa$ curves connecting the boundary points x_{a_j} and x_{b_j} for $j \in \{1, 2, \dots, N\}$ according to the connectivity α . Denote by \mathbb{E}_α the expectation with respect to \mathbb{P}_α . Define \mathbb{Q}_α to be the measure which is absolutely continuous with respect to \mathbb{P}_α with Radon-Nikodym derivative

$$\frac{d\mathbb{Q}_\alpha}{d\mathbb{P}_\alpha} = R_\alpha(\Omega; \eta_1, \dots, \eta_N) := \mathbb{1}_{\{\eta_j \cap \eta_k = \emptyset \ \forall \ j \neq k\}} \exp(cm_\alpha(\Omega; \eta_1, \dots, \eta_N)), \quad (3.3)$$

where $c = (3\kappa - 8)(6 - \kappa)/2\kappa$.

First, we prove that the total mass $|\mathbb{Q}_\alpha| = \mathbb{E}_\alpha[R_\alpha(\Omega; \eta_1, \dots, \eta_N)]$ of \mathbb{Q}_α is positive and finite. Positivity is clear from the definition (3.3). We prove the finiteness by induction on N , using the cascade property of Lemma 3.1. The base case $N = 1$ is obvious: $R_\alpha = 1$. Let $N \geq 2$ and assume that $|\mathbb{Q}_{\hat{\alpha}}|$ is finite for all $\hat{\alpha} \in \text{LP}_{N-1}$. Using Lemma 3.1, we write the Radon-Nikodym derivative (3.3) in the form

$$R_\alpha(\Omega; \eta_1, \dots, \eta_N) = R_{\hat{\alpha}}(\Omega; \eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_N) \times \mathbb{1}_{\{\eta_j \subset \hat{\Omega}_j\}} \exp(c\mu(\Omega; \eta_j, \Omega \setminus \hat{\Omega}_j)), \quad (3.4)$$

for a fixed $j \in \{1, \dots, N\}$, where $\hat{\alpha} = \alpha / \{a_j, b_j\}$. Thus, we have

$$\begin{aligned} \mathbb{E}_\alpha[R_\alpha(\Omega; \eta_1, \dots, \eta_N)] &= \mathbb{E}_\alpha[\mathbb{E}_\alpha[R_\alpha(\Omega; \eta_1, \dots, \eta_N) \mid \eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_N]] \\ &= \mathbb{E}_{\hat{\alpha}}\left[R_{\hat{\alpha}}(\Omega; \eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_N) \left(\frac{H_{\hat{\Omega}_j}(x_{a_j}, x_{b_j})}{H_\Omega(x_{a_j}, x_{b_j})}\right)^h\right] \quad [\text{by Lemma 2.2}] \\ &\leq \mathbb{E}_{\hat{\alpha}}[R_{\hat{\alpha}}(\Omega; \eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_N)] \quad [\text{by (2.3)}] \\ &\leq 1. \quad [\text{by ind. hypothesis}] \end{aligned}$$

Noting that the Radon-Nikodym derivative (3.3) also depends on the fixed boundary points x_1, \dots, x_{2N} , we define the function f_α of $2N$ complex variables $x_1, \dots, x_{2N} \in \partial\Omega$ by

$$f_\alpha(\Omega; x_1, \dots, x_{2N}) := \mathbb{E}_\alpha[R_\alpha(\Omega; \eta_1, \dots, \eta_N)] = |\mathbb{Q}_\alpha|. \quad (3.5)$$

Note that f_α is a conformal invariant. From the above analysis, we see that it is also bounded:

$$0 < f_\alpha \leq 1. \quad (3.6)$$

Second, we show that, for each $j \in \{1, \dots, N\}$, under the probability measure $\mathbb{Q}_\alpha^\# := \mathbb{Q}_\alpha / |\mathbb{Q}_\alpha|$, the conditional law of η_j given $\{\eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_N\}$ is the chordal SLE $_\kappa$ connecting x_{a_j} and x_{b_j} in the domain $\hat{\Omega}_j$. By the cascade property (3.4), given $\{\eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_N\}$, the conditional law of η_j is the same as $\mathbb{P}(\Omega; x_{a_j}, x_{b_j})$ weighted by $\mathbb{1}_{\{\eta_j \subset \hat{\Omega}_j\}} \exp(c\mu(\Omega; \eta_j, \Omega \setminus \hat{\Omega}_j))$. Now, by Lemma 2.2, this is the same as the law of the chordal SLE $_\kappa$ in $\hat{\Omega}_j$ connecting x_{a_j} and x_{b_j} . This completes the proof. \square

3.2 Properties of Global Multiple SLEs

Let $\kappa \in (0, 4]$ and $h = (6 - \kappa)/2\kappa$. We set $\mathcal{Z}_\emptyset := 1$, and for all integers $N \geq 1$ and for all link patterns $\alpha = \{\{a_1, b_1\}, \dots, \{a_N, b_N\}\} \in \text{LP}_N$, we define

$$\mathcal{Z}_\alpha: \mathfrak{X}_{2N} \rightarrow \mathbb{R}_{>0}, \quad \mathcal{Z}_\alpha(x_1, \dots, x_{2N}) := f_\alpha(\mathbb{H}; x_1, \dots, x_{2N}) \times \prod_{j=1}^N H_{\mathbb{H}}(x_{a_j}, x_{b_j})^h, \quad (3.7)$$

where $f_\alpha = |\mathbb{Q}_\alpha|$ is the function defined in (3.5) and $H_{\mathbb{H}}$ is the boundary Poisson kernel in the upper half-plane \mathbb{H} . More generally, for a simply connected domain Ω with $x_1, \dots, x_{2N} \in \partial\Omega$ in counterclockwise order on analytic boundary segments, we define

$$\mathcal{Z}_\alpha(\Omega; x_1, \dots, x_{2N}) := f_\alpha(\Omega; x_1, \dots, x_{2N}) \times \prod_{j=1}^N H_\Omega(x_{a_j}, x_{b_j})^h. \quad (3.8)$$

This definition agrees with (1.5), by the conformal covariance of the boundary Poisson kernel H_Ω and the conformal invariance of f_α .

3.2.1 Boundary Perturbation Property

Multiple SLEs have a boundary perturbation property analogous to Lemma 2.2. To state it, we use the specific notation $\mathbb{Q}_\alpha^\#(\Omega; x_1, \dots, x_{2N})$ for the global N -SLE $_\kappa$ probability measure associated to the link pattern α in the domain Ω with marked points $x_1, \dots, x_{2N} \in \partial\Omega$.

Proposition 3.4. *Let Ω be a simply connected domain with non-polar boundary, and let $x_1, \dots, x_{2N} \in \partial\Omega$ be distinct boundary points on analytic boundary segments. Assume that $\hat{\Omega} \subset \Omega$ is simply connected and that $\hat{\Omega}$ and Ω agree in neighborhoods of x_1, \dots, x_{2N} . Then, $\mathbb{Q}_\alpha^\#(\hat{\Omega}; x_1, \dots, x_{2N})$ is absolutely continuous with respect to $\mathbb{Q}_\alpha^\#(\Omega; x_1, \dots, x_{2N})$ with Radon-Nikodym derivative given by*

$$\frac{d\mathbb{Q}_\alpha^\#(\hat{\Omega}; x_1, \dots, x_{2N})}{d\mathbb{Q}_\alpha^\#(\Omega; x_1, \dots, x_{2N})} = \frac{\mathcal{Z}_\alpha(\Omega; x_1, \dots, x_{2N})}{\mathcal{Z}_\alpha(\hat{\Omega}; x_1, \dots, x_{2N})} \times \mathbb{1}_{\{\eta_j \subset \hat{\Omega} \vee j\}} \times \exp\left(c\mu\left(\Omega; \Omega \setminus \hat{\Omega}, \bigcup_{j=1}^N \eta_j\right)\right),$$

where $(\eta_1, \dots, \eta_N) \sim \mathbb{Q}_\alpha^\#(\Omega; x_1, \dots, x_{2N})$. Moreover, when $\kappa \leq 8/3$, we have

$$\mathcal{Z}_\alpha(\Omega; x_1, \dots, x_{2N}) \geq \mathcal{Z}_\alpha(\hat{\Omega}; x_1, \dots, x_{2N}). \quad (3.9)$$

Proof. From (3.3) and Lemma 3.2, we see that

$$\begin{aligned} & \mathbb{1}_{\{\eta_j \subset \hat{\Omega} \vee j\}} d\mathbb{Q}_\alpha(\Omega; x_1, \dots, x_{2N}) \\ &= \mathbb{1}_{\{\eta_j \subset \hat{\Omega} \vee j\}} \mathbb{1}_{\{\eta_j \cap \eta_k = \emptyset \vee j \neq k\}} \exp(cm_\alpha(\Omega; \eta_1, \dots, \eta_N)) d\mathbb{P}_\alpha \\ &= \mathbb{1}_{\{\eta_j \subset \hat{\Omega} \vee j\}} \mathbb{1}_{\{\eta_j \cap \eta_k = \emptyset \vee j \neq k\}} \exp(cm_\alpha(\hat{\Omega}; \eta_1, \dots, \eta_N)) \times \exp\left(-c\mu\left(\Omega; \Omega \setminus \hat{\Omega}, \bigcup_{j=1}^N \eta_j\right)\right) \\ & \quad \times \prod_{j=1}^N \exp\left(c\mu\left(\Omega; \Omega \setminus \hat{\Omega}, \eta_j\right)\right) d\mathbb{P}(\Omega; x_{a_j}, x_{b_j}) \end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned}
& \mathbb{1}_{\{\eta_j \subset \hat{\Omega} \forall j\}} d\mathbb{Q}_\alpha(\Omega; x_1, \dots, x_{2N}) \\
&= \mathbb{1}_{\{\eta_j \cap \eta_k = \emptyset \forall j \neq k\}} \exp(cm_\alpha(\hat{\Omega}; \eta_1, \dots, \eta_N)) \times \exp\left(-c\mu\left(\Omega; \Omega \setminus \hat{\Omega}, \bigcup_{j=1}^N \eta_j\right)\right) \\
&\quad \times \prod_{j=1}^N \left(\frac{H_{\hat{\Omega}}(x_{a_j}, x_{b_j})}{H_\Omega(x_{a_j}, x_{b_j})}\right)^h d\mathbb{P}(\hat{\Omega}; x_{a_j}, x_{b_j}) \\
&= \exp\left(-c\mu\left(\Omega; \Omega \setminus \hat{\Omega}, \bigcup_{j=1}^N \eta_j\right)\right) \prod_{j=1}^N \left(\frac{H_{\hat{\Omega}}(x_{a_j}, x_{b_j})}{H_\Omega(x_{a_j}, x_{b_j})}\right)^h d\mathbb{Q}_\alpha(\hat{\Omega}; x_1, \dots, x_{2N}).
\end{aligned}$$

Combining with the definition (3.8), we obtain the asserted Radon-Nikodym derivative. The monotonicity property (3.9) follows from the fact that when $\kappa \leq 8/3$, we have $c \leq 0$ and thus,

$$1 \geq \mathbb{P}[\eta_j \subset \hat{\Omega} \forall j] \geq \frac{\mathcal{Z}_\alpha(\hat{\Omega}; x_1, \dots, x_{2N})}{\mathcal{Z}_\alpha(\Omega; x_1, \dots, x_{2N})}.$$

□

3.2.2 Marginal Law

Next we prove a cascade property for the measure $\mathbb{Q}_\alpha^\#$. Fix $\alpha = \{\{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_N, b_N\}\} \in \text{LP}_N$ and $j \in \{1, \dots, N\}$. Let η_j be the curve connecting x_{a_j} and x_{b_j} in the global N -SLE $_\kappa$ with law $\mathbb{Q}_\alpha^\#$, as in Theorem 1.3. We may assume that $a_j < b_j$. The domain $\Omega \setminus \eta_j$ has two connected components, which we denote by D_j^L and D_j^R . The link $\{a_j, b_j\}$ divides also the link pattern α into two sub-link patterns, connecting $\{b_j+1, \dots, a_j-1\}$ and $\{a_j+1, \dots, b_j-1\}$, respectively. After relabeling the remaining indices, we denote these link patterns by α_j^L and α_j^R . The notations are illustrated in Figure 3.2.

Proposition 3.5. *The marginal law of η_j under $\mathbb{Q}_\alpha^\#$ is absolutely continuous with respect to the law $\mathbb{P}(\Omega; x_{a_j}, x_{b_j})$ of the chordal SLE $_\kappa$ connecting x_{a_j} and x_{b_j} , with Radon-Nikodym derivative given by*

$$\frac{H_\Omega(x_{a_j}, x_{b_j})^h}{\mathcal{Z}_\alpha(\Omega; x_1, \dots, x_{2N})} \times \mathcal{Z}_{\alpha_j^L}(D_j^L; x_{b_j+1}, x_{b_j+2}, \dots, x_{a_j-1}) \times \mathcal{Z}_{\alpha_j^R}(D_j^R; x_{a_j+1}, x_{a_j+2}, \dots, x_{b_j-1}).$$

Proof. Note that the points $x_{b_j+1}, \dots, x_{a_j-1}$ (resp. $x_{a_j+1}, \dots, x_{b_j-1}$) are along the boundary of D_j^L (resp. D_j^R) in counterclockwise order. Denote by $(\eta_1, \dots, \eta_N) \in X_0^\alpha(\Omega; x_1, \dots, x_{2N})$ the global N -SLE $_\kappa$ with law $\mathbb{Q}_\alpha^\#$. Amongst the curves $\eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_N$, we denote by $\eta_1^L, \dots, \eta_l^L$ the ones contained in D_j^L and by $\eta_1^R, \dots, \eta_r^R$ the ones contained in D_j^R (so $l + r = N - 1$).

First, we prove by induction on N that

$$m_\alpha(\Omega; \eta_1, \dots, \eta_N) = m_{\alpha_j^L}(D_j^L; \eta_1^L, \dots, \eta_l^L) + m_{\alpha_j^R}(D_j^R; \eta_1^R, \dots, \eta_r^R) + \sum_{k \neq j} \mu(\Omega; \eta_j, \eta_k). \quad (3.10)$$

Equation (3.10) trivially holds for $N = 1$, since $m_\emptyset = 0 = m_\alpha$. By symmetry, we may assume that $\{a_j, b_j\} \neq \{2N-1, 2N\}$, and $\{2N-1, 2N\} \in \alpha \cap \alpha_j^L$. Suppose $\eta_1 = \eta_1^L \subset D_j^L$ is the curve connecting x_{2N-1} and x_{2N} . Denote by $\hat{\alpha} = \alpha / \{2N-1, 2N\}$, and define $\hat{\alpha}_j^L$ and $\hat{\alpha}_j^R$ for $\hat{\alpha}$ as above. Then $\alpha_j^L = \hat{\alpha}_j^L \cup \{\{2N-1, 2N\}\}$ and $\hat{\alpha}_j^R = \alpha_j^R$. An application of Lemma 3.1 and the induction hypothesis gives

$$\begin{aligned}
m_\alpha(\Omega; \eta_1, \dots, \eta_N) &= m_{\hat{\alpha}}(\Omega; \eta_2, \dots, \eta_N) + \mu(\Omega; \eta_1, \Omega \setminus \hat{\Omega}_1) \\
&= m_{\hat{\alpha}_j^L}(D_j^L; \eta_2^L, \dots, \eta_l^L) + m_{\alpha_j^R}(D_j^R; \eta_1^R, \dots, \eta_r^R) + \sum_{k \neq 1, j} \mu(\Omega; \eta_j, \eta_k) + \mu(\Omega; \eta_1, \Omega \setminus \hat{\Omega}_1).
\end{aligned}$$

Combining with the decomposition $\mu(\Omega; \eta_1, \Omega \setminus \hat{\Omega}_1) = \mu(D_j^L; \eta_1, D_j^L \setminus \hat{\Omega}_1) + \mu(\Omega; \eta_1, \eta_j)$, we obtain

$$\begin{aligned} m_\alpha(\Omega; \eta_1, \dots, \eta_N) &= m_{\hat{\alpha}_j^L}(D_j^L; \eta_2^L, \dots, \eta_l^L) + \mu(D_j^L; \eta_1, D_j^L \setminus \hat{\Omega}_1) + m_{\alpha_j^R}(D_j^R; \eta_1^R, \dots, \eta_r^R) + \sum_{k \neq j} \mu(\Omega; \eta_j, \eta_k) \\ &= m_{\alpha_j^L}(D_j^L; \eta_1^L, \dots, \eta_l^L) + m_{\alpha_j^R}(D_j^R; \eta_1^R, \dots, \eta_r^R) + \sum_{k \neq j} \mu(\Omega; \eta_j, \eta_k), \end{aligned}$$

by Lemma 3.1. This completes the proof of (3.10).

Next, we prove the conclusion. From (3.3), we see that

$$\begin{aligned} d\mathbb{Q}_\alpha &= \mathbb{1}_{\{\eta_i \cap \eta_k = \emptyset \ \forall \ i \neq k\}} \exp(cm_\alpha(\Omega; \eta_1, \dots, \eta_N)) \prod_{k=1}^N d\mathbb{P}(\Omega; x_{a_k}, x_{b_k}) \\ &= \mathbb{1}_{\{\eta_i \cap \eta_k = \emptyset \ \forall \ i \neq k\}} \exp\left(cm_{\alpha_j^L}(D_j^L; \eta_1^L, \dots, \eta_l^L)\right) \times \exp\left(cm_{\alpha_j^R}(D_j^R; \eta_1^R, \dots, \eta_r^R)\right) \\ &\quad \times \prod_{k \neq j} \exp(c\mu(\Omega; \eta_j, \eta_k)) d\mathbb{P}(\Omega; x_{a_k}, x_{b_k}) \times d\mathbb{P}(\Omega; x_{a_j}, x_{b_j}) \quad [\text{by (3.10)}] \\ &= \mathbb{1}_{\{\eta_i \cap \eta_k = \emptyset \ \forall \ i \neq k\}} \times d\mathbb{P}(\Omega; x_{a_j}, x_{b_j}) \quad [\text{by Lemma 2.2}] \\ &\quad \times \exp\left(cm_{\alpha_j^L}(D_j^L; \eta_1^L, \dots, \eta_l^L)\right) \times \prod_{\{a,b\} \in \alpha_j^L} \left(\frac{H_{D_j^L}(x_a, x_b)}{H_\Omega(x_a, x_b)}\right)^h d\mathbb{P}(D_j^L; x_a, x_b) \\ &\quad \times \exp\left(cm_{\alpha_j^R}(D_j^R; \eta_1^R, \dots, \eta_r^R)\right) \times \prod_{\{a,b\} \in \alpha_j^R} \left(\frac{H_{D_j^R}(x_a, x_b)}{H_\Omega(x_a, x_b)}\right)^h d\mathbb{P}(D_j^R; x_a, x_b). \end{aligned}$$

By definitions (3.5), (3.7), and (3.8), this implies that the law of η_j under $\mathbb{Q}_\alpha^\# = \mathbb{Q}_\alpha/f_\alpha$ is absolutely continuous with respect to $\mathbb{P}(\Omega; x_{a_j}, x_{b_j})$ and the Radon-Nikodym derivative has the asserted form. \square

Corollary 3.6. *Let $\alpha = \{\{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_N, b_N\}\} \in \text{LP}_N$ and $j \in \{1, \dots, 2N-1\}$ such that $\{j, j+1\} \in \alpha$, and denote $\hat{\alpha} = \alpha/\{j, j+1\}$. Let η_j be the curve connecting x_j and x_{j+1} in the global N -SLE $_\kappa$ with law $\mathbb{Q}_\alpha^\#$. Denote by D_j the connected component of $\Omega \setminus \eta_j$ having $x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N}$ on its boundary. Then the marginal law of η_j under $\mathbb{Q}_\alpha^\#$ is absolutely continuous with respect to the law $\mathbb{P}(\Omega; x_j, x_{j+1})$ of the chordal SLE $_\kappa$ connecting x_j and x_{j+1} , with Radon-Nikodym derivative given by*

$$\frac{H_\Omega(x_j, x_{j+1})^h}{Z_\alpha(\Omega; x_1, \dots, x_{2N})} \times \mathcal{Z}_{\hat{\alpha}}(D_j; x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N}).$$

4 Pure Partition Functions for Multiple SLE $_\kappa$ with $\kappa \in (0, 4]$

In this section, we prove Theorem 1.1, which says that the pure partition functions of multiple SLEs are smooth, positive, and (essentially) unique. Corollary 1.2 in Section 4.3 relates them to certain extremal multiple SLE measures, thus verifying a conjecture from [BBK05, KP16]. In Section 4.3, we also complete the proof of Theorem 1.3, by proving in Lemma 4.15 that the local and global SLE $_\kappa$ associated to α agree.

4.1 Proof of Theorem 1.1

We prove Theorem 1.1 after a succession of lemmas establishing the asserted properties of the pure partition functions \mathcal{Z}_α . From the Brownian loop measure construction, it is difficult to show directly that the partition function \mathcal{Z}_α defined in (3.7) is a solution to the system (PDE) (1.1), because it is not clear why \mathcal{Z}_α should be twice continuously differentiable. To this end, we use the hypoellipticity of the PDEs (1.1) from the work of J. Dubédat, stated in Proposition 2.5. With the hypoellipticity, it suffices to prove that \mathcal{Z}_α is a distributional solution to (PDE) (1.1), which we establish in Lemma 4.4 by constructing a local martingale from the conditional expectation of the Radon-Nikodym derivative (3.3).

Lemma 4.1. *The function \mathcal{Z}_α defined in (3.7) satisfies the bound (1.4).*

Proof. The asserted bound (1.4) follows immediately from the bound $0 < f_\alpha \leq 1$ established in (3.6). \square

Lemma 4.2. *The function \mathcal{Z}_α defined in (3.7) satisfies the Möbius covariance (COV) (1.2).*

Proof. The function $f_\alpha(\mathbb{H}; x_1, \dots, x_{2N})$ is Möbius invariant by (3.3). Combining with the conformal covariance of the boundary Poisson kernel (2.1), we see that \mathcal{Z}_α satisfies the Möbius covariance (COV) (1.2). \square

Lemma 4.3. *The function \mathcal{Z}_α defined in (3.7) satisfies the asymptotics property (ASY) (1.3).*

Proof. Let $\alpha = \{\{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_N, b_N\}\} \in \text{LP}_N$. Without loss of generality, we let $j = 1 = a_1$ and we denote its pair in α by $k = b_1$. We also denote $\hat{\alpha} = \{\{a_2, b_2\}, \dots, \{a_N, b_N\}\} \in \text{LP}_{N-1}$ as in the asserted asymptotics property (ASY) (1.3). Let η_1 be the curve connecting x_1 and x_k , and denote by $\hat{\Omega}_1$ the connected component of $\mathbb{H} \setminus \{\eta_2, \dots, \eta_N\}$ that has x_1 and x_k on its boundary.

As in the proof of Proposition 3.3, using Lemma 3.1 and Lemma 2.2, we write

$$\begin{aligned} \mathbb{E}_\alpha [R_\alpha(\mathbb{H}; \eta_1, \eta_2, \dots, \eta_N)] &= \mathbb{E}_\alpha \left[R_{\hat{\alpha}}(\mathbb{H}; \eta_2, \dots, \eta_N) \mathbb{1}_{\{\eta_1 \subset \hat{\Omega}_1\}} \exp(c\mu(\mathbb{H}; \eta_1, \mathbb{H} \setminus \hat{\Omega}_1)) \right] \\ &= \mathbb{E}_{\hat{\alpha}} \left[R_{\hat{\alpha}}(\mathbb{H}; \eta_2, \dots, \eta_N) \left(\frac{H_{\hat{\Omega}_1}(x_1, x_k)}{H_{\mathbb{H}}(x_1, x_k)} \right)^h \right] \\ &= \mathbb{E}_{\hat{\alpha}} \left[R_{\hat{\alpha}}(\mathbb{H}; \eta_2, \dots, \eta_N) (\mu_{\mathbb{H}}^\#(x_1, x_k) [\mathcal{E} \subset \hat{\Omega}_1])^h \right], \end{aligned} \quad (4.1)$$

where we recognized the ratio of Poisson kernels as the Brownian excursion measure $\mu_{\mathbb{H}}^\#(x_1, x_k) [\mathcal{E} \subset \hat{\Omega}_1]$ in the upper half-plane for Brownian paths \mathcal{E} connecting x_1 and x_k in $\hat{\Omega}_1$ — see [LW04, Section 3.3] for details. Now, we notice that, as $x_1, x_2 \rightarrow \xi$ for $\xi < x_3$, we almost surely have

$$\mu_{\mathbb{H}}^\#(x_1, x_k) [\mathcal{E} \subset \hat{\Omega}_1] \longrightarrow \begin{cases} 0 & \text{if } \{1, 2\} \notin \alpha, \text{ that is, } k \neq 2 \\ 1 & \text{if } \{1, 2\} \in \alpha, \text{ that is, } k = 2. \end{cases} \quad (4.2)$$

Combining (4.1) and (4.2), and using the Dominated Convergence Theorem, we get the asserted asymptotics property (ASY) (1.3) for the function \mathcal{Z}_α defined in (3.7) :

$$\begin{aligned} &\lim_{x_1, x_2 \rightarrow \xi} \frac{\mathcal{Z}_\alpha(x_1, \dots, x_{2N})}{(x_2 - x_1)^{-2h}} \\ &= \prod_{j=2}^N H_{\mathbb{H}}(x_{a_j}, x_{b_j})^h \times \lim_{x_1, x_2 \rightarrow \xi} \mathbb{E}_{\hat{\alpha}} \left[R_{\hat{\alpha}}(\mathbb{H}; \eta_2, \dots, \eta_N) (\mu_{\mathbb{H}}^\#(x_1, x_k) [\mathcal{E} \subset \hat{\Omega}_1])^h \right] \\ &= \begin{cases} 0 & \text{if } \{1, 2\} \notin \alpha, \text{ that is, } k \neq 2 \\ \mathcal{Z}_{\hat{\alpha}}(x_3, \dots, x_{2N}) & \text{if } \{1, 2\} \in \alpha, \text{ that is, } k = 2. \end{cases} \end{aligned}$$

This finishes the proof. \square

Lemma 4.4. *The function \mathcal{Z}_α defined in (3.7) is smooth and it satisfies the system (PDE) (1.1) of $2N$ partial differential equations of second order.*

Proof. Let $\alpha = \{\{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_N, b_N\}\} \in \text{LP}_N$. We prove that \mathcal{Z}_α satisfies the partial differential equation of (1.1) for $i = 1$; the other cases follow by symmetry. Denote the pair of $i = 1 = a_1$ in α by $k = b_1$, and denote $\hat{\alpha} = \{\{a_2, b_2\}, \dots, \{a_N, b_N\}\} \in \text{LP}_{N-1}$. Let η_1 be the curve connecting x_1 and x_k , and denote by $\hat{\Omega}_1$ the connected component of $\mathbb{H} \setminus \{\eta_2, \dots, \eta_N\}$ that has x_1 and x_k on its boundary. Then, given $\{\eta_2, \dots, \eta_N\}$, the conditional law of η_1 is that of the chordal SLE_κ in $\hat{\Omega}_1$ from x_1 to x_k .

Recall from (3.7) that the function \mathcal{Z}_α is defined in terms of the expectation of R_α . We calculate the conditional expectation $\mathbb{E}_\alpha[R_\alpha(\mathbb{H}; \eta_1, \dots, \eta_N) | \eta_1[0, t]]$ for small $t > 0$, and construct a local martingale involving the function \mathcal{Z}_α . Diffusion theory then provides us with the desired partial differential equation (1.1) in distributional sense, and we may conclude by hypoellipticity (Proposition 2.5).

Given $\eta_1[0, t]$, set $K_t := \eta_1[0, t]$ and $H_t := \mathbb{H} \setminus K_t$ and $\tilde{\eta}_1 := (\eta_1(s), s \geq t)$. Using the observation that the Brownian loop measure can be decomposed as

$$\mu(\mathbb{H}; \eta_1, \mathbb{H} \setminus \hat{\Omega}_1) = \mu(H_t; \tilde{\eta}_1, \mathbb{H} \setminus \hat{\Omega}_1) + \mu(\mathbb{H}; K_t, \mathbb{H} \setminus \hat{\Omega}_1), \quad (4.3)$$

with Lemmas 3.1 and 3.2, we write the quantity m_α defined in (3.1) in the following form:

$$\begin{aligned} m_\alpha(\mathbb{H}; \eta_1, \eta_2, \dots, \eta_N) &= m_{\hat{\alpha}}(\mathbb{H}; \eta_2, \dots, \eta_N) + \mu(\mathbb{H}; \eta_1, \mathbb{H} \setminus \hat{\Omega}_1) && [\text{by Lemma 3.1}] \\ &= m_{\hat{\alpha}}(H_t; \eta_2, \dots, \eta_N) + \sum_{j=2}^N \mu(\mathbb{H}; K_t, \eta_j) - \mu(\mathbb{H}; K_t, \bigcup_{j=2}^N \eta_j) && [\text{by Lemma 3.2}] \\ &\quad + \mu(H_t; \tilde{\eta}_1, \mathbb{H} \setminus \hat{\Omega}_1) + \mu(\mathbb{H}; K_t, \mathbb{H} \setminus \hat{\Omega}_1). && [\text{by (4.3)}] \end{aligned}$$

Note that $\mu(\mathbb{H}; K_t, \bigcup_{j=2}^N \eta_j) = \mu(\mathbb{H}; K_t, \mathbb{H} \setminus \hat{\Omega}_1)$, so the last terms of the last two lines cancel. Combining the first terms of these two lines with the help of Lemma 3.1, we obtain

$$m_\alpha(\mathbb{H}; \eta_1, \eta_2, \dots, \eta_N) = m_\alpha(H_t; \tilde{\eta}_1, \eta_2, \dots, \eta_N) + \sum_{j=2}^N \mu(\mathbb{H}; K_t, \eta_j).$$

Using this, we write the Radon-Nikodym derivative (3.3) in the form

$$\begin{aligned} R_\alpha(\mathbb{H}; \eta_1, \eta_2, \dots, \eta_N) &= \mathbb{1}_{\{\eta_j \cap \eta_k = \emptyset \ \forall \ j \neq k\}} \exp(cm_\alpha(H_t; \tilde{\eta}_1, \eta_2, \dots, \eta_N)) \times \prod_{j=2}^N \exp(c\mu(\mathbb{H}; K_t, \eta_j)) \\ &= R_\alpha(H_t; \tilde{\eta}_1, \eta_2, \dots, \eta_N) \times \prod_{j=2}^N \mathbb{1}_{\{\eta_j \subset H_t\}} \exp(c\mu(\mathbb{H}; K_t, \eta_j)) && [\text{by (3.3)}] \\ &= R_\alpha(H_t; \tilde{\eta}_1, \eta_2, \dots, \eta_N) \times \prod_{j=2}^N \left(\frac{H_{H_t}(x_{a_j}, x_{b_j})}{H_{\mathbb{H}}(x_{a_j}, x_{b_j})} \right)^h \frac{d\mathbb{P}(H_t; x_{a_j}, x_{b_j})}{d\mathbb{P}(\mathbb{H}; x_{a_j}, x_{b_j})}. && [\text{by Lemma 2.2}] \end{aligned}$$

This implies that, given $K_t = \eta_1[0, t]$, the conditional expectation of R_α is

$$\begin{aligned} \mathbb{E}_\alpha[R_\alpha(\mathbb{H}; \eta_1, \eta_2, \dots, \eta_N) | K_t] &= \mathbb{E}_\alpha[R_\alpha(H_t; \tilde{\eta}_1, \eta_2, \dots, \eta_N)] \times \prod_{j=2}^N \left(\frac{H_{H_t}(x_{a_j}, x_{b_j})}{H_{\mathbb{H}}(x_{a_j}, x_{b_j})} \right)^h \\ &= f_\alpha(H_t; \eta_1(t), x_2, \dots, x_{2N}) \times \prod_{j=2}^N \left(\frac{H_{H_t}(x_{a_j}, x_{b_j})}{H_{\mathbb{H}}(x_{a_j}, x_{b_j})} \right)^h. \end{aligned}$$

Let g_t denote the Loewner map associated to η_1 , normalized at ∞ . Since f_α is a conformal invariant, we can write, using Equation (3.7) and the formula $H_{\mathbb{H}}(x, y) = |y - x|^{-2}$ for the Poisson kernel in \mathbb{H} ,

$$\begin{aligned} f_\alpha(H_t; \eta_1(t), x_2, \dots, x_{2N}) &= f_\alpha(\mathbb{H}; W_t, g_t(x_2), \dots, g_t(x_{2N})) \\ &= (g_t(x_k) - W_t)^{2h} \prod_{j=2}^N (g_t(x_{b_j}) - g_t(x_{a_j}))^{2h} \times \mathcal{Z}_\alpha(W_t, g_t(x_2), \dots, g_t(x_{2N})). \end{aligned}$$

On the other hand, by (2.1), we have

$$\prod_{j=2}^N H_{H_t}(x_{a_j}, x_{b_j})^h = \prod_{j=2}^N g'_t(x_{a_j})^h g'_t(x_{b_j})^h (g_t(x_{b_j}) - g_t(x_{a_j}))^{-2h}.$$

Combining the above observations, we get

$$\mathbb{E}_\alpha[R_\alpha(\mathbb{H}; \eta_1, \eta_2, \dots, \eta_N) | K_t] = \prod_{j=1}^N (x_{b_j} - x_{a_j})^{2h} \times \frac{M_t}{N_t},$$

where

$$N_t := g'_t(x_k)^h (g_t(x_k) - W_t)^{-2h}, \quad M_t := \prod_{j=2}^{2N} g'_t(x_j)^h \times \mathcal{Z}_\alpha(W_t, g_t(x_2), \dots, g_t(x_{2N})).$$

Thus, M_t/N_t is a local martingale for η_1 . Let γ be the chordal SLE_κ in \mathbb{H} from x_1 to ∞ . By Lemma 2.1, the law of η_1 is the same as the law of γ weighted by N_t . Therefore, M_t is a local martingale for γ .

Now, we have $M_t = F(X_t)$, where

$$F(x_1, \dots, x_{2N}, y_2, \dots, y_{2N}) = \prod_{j=2}^{2N} y_j^h \times \mathcal{Z}_\alpha(x_1, x_2, \dots, x_{2N})$$

is a continuous function and $X_t = (W_t, g_t(x_1), \dots, g_t(x_{2N}), g'_t(x_2), \dots, g'_t(x_{2N}))$ is a Feller process with infinitesimal generator

$$A = \frac{\kappa}{2} \partial_1^2 + \sum_{j=2}^{2N} \left(\frac{2}{x_j - x_1} \partial_j - \frac{2y_j}{(x_j - x_1)^2} \partial_{2N-1+j} \right).$$

By [RY94, Chapter VII, Proposition 1.7], because $M_t = F(X_t)$ is a local martingale, F belongs to the domain of A in distributional sense, and we have $AF = 0$. A calculation shows that the equation $AF = 0$ is equivalent to the property that \mathcal{Z}_α satisfies the PDE (1.1) for $i = 1$ in distributional sense. It now follows from Proposition 2.5 that \mathcal{Z}_α is a smooth solution to the asserted PDE for $i = 1$. \square

The next lemma is a crucial tool in our proof of uniqueness of the pure partition functions. It is an immediate consequence of the deep result [FK15b, Lemma 1] that we stated as Theorem 2.3 in Section 2.3. Recall that \mathcal{S}_N denotes the solution space defined in (2.8).

Lemma 4.5. *Let $\{F_\alpha: \alpha \in \text{LP}\}$ be a collection of functions $F_\alpha \in \mathcal{S}_N$, for $\alpha \in \text{LP}_N$, satisfying (ASY) (1.3) with normalization $F_\emptyset = 1$. Then, the collection $\{F_\alpha: \alpha \in \text{LP}\}$ is unique.*

Proof. Let $\{F_\alpha: \alpha \in \text{LP}\}$ and $\{\tilde{F}_\alpha: \alpha \in \text{LP}\}$ be two collections satisfying the properties listed in the assertion. Then, for any $\alpha \in \text{LP}_N$, the difference $F_\alpha - \tilde{F}_\alpha$ has the asymptotics property

$$\lim_{x_j, x_{j+1} \rightarrow \xi} \frac{F_\alpha(x_1, \dots, x_{2N})}{(x_{j+1} - x_j)^{-2h}} = 0 \quad \text{for all } j \in \{2, \dots, 2N-1\} \text{ and } \xi \in (x_{j-1}, x_{j+2}),$$

so Theorem 2.3 shows that $F_\alpha - \tilde{F}_\alpha \equiv 0$. The asserted uniqueness follows. \square

We are now ready to conclude:

Theorem 1.1. *Let $\kappa \in (0, 4]$. There exists a unique collection $\{\mathcal{Z}_\alpha: \alpha \in \text{LP}\}$ of smooth functions $\mathcal{Z}_\alpha: \mathfrak{X}_{2N} \rightarrow \mathbb{R}_{>0}$, for $\alpha \in \text{LP}_N$, satisfying the normalization $\mathcal{Z}_\emptyset = 1$ and properties (PDE) (1.1), (COV) (1.2), (ASY) (1.3), and, for all $\alpha = \{\{a_1, b_1\}, \dots, \{a_N, b_N\}\} \in \text{LP}_N$, the power law bound*

$$0 < \mathcal{Z}_\alpha(x_1, \dots, x_{2N}) \leq \prod_{j=1}^N |x_{b_j} - x_{a_j}|^{-2h}, \quad \text{where } h = \frac{6 - \kappa}{2\kappa}. \quad (1.4)$$

Proof. The functions $\{\mathcal{Z}_\alpha: \alpha \in \text{LP}\}$ defined in (3.7) satisfy all the asserted properties: (PDE) (1.1), (COV) (1.2), and (ASY) (1.3) respectively follow from Lemmas 4.4, 4.2, and 4.3, and the bound (1.4) is the content of Lemma 4.1. Uniqueness follows from Lemma 4.5, since the bound (1.4) implies (2.7). \square

In fact, the pure partition functions of Theorem 1.1 are linearly independent.

Proposition 4.6. *Let $\{\mathcal{L}_\alpha: \alpha \in \text{LP}\}$ be the collection of linear functionals defined in (2.11) and let $\{\mathcal{Z}_\alpha: \alpha \in \text{LP}\}$ be the collection of functions of Theorem 1.1. Then, we have $\mathcal{Z}_\alpha \in \mathcal{S}_N$ and*

$$\mathcal{L}_\alpha(\mathcal{Z}_\beta) = \delta_{\alpha,\beta} = \begin{cases} 1 & \text{if } \beta = \alpha \\ 0 & \text{if } \beta \neq \alpha, \end{cases} \quad (4.4)$$

for all $\alpha, \beta \in \text{LP}_N$. In particular, the set $\{\mathcal{Z}_\alpha: \alpha \in \text{LP}_N\}$ is linearly independent and it thus forms a basis for the C_N -dimensional solution space \mathcal{S}_N with dual basis $\{\mathcal{L}_\alpha: \alpha \in \text{LP}_N\}$.

Proof. Theorem 1.1 shows that we have $\mathcal{Z}_\alpha \in \mathcal{S}_N$. Property (4.4) follows from the asymptotics properties (ASY) (1.3) of the functions \mathcal{Z}_α from Lemma 4.3, and the last assertion follows immediately from this. \square

In [KP16, Theorem 4.1], K. Kytölä and E. Peltola constructed candidates for the pure partition functions \mathcal{Z}_α with $\kappa \in (0, 8) \setminus \mathbb{Q}$ using Coulomb gas techniques and a hidden quantum group symmetry on the solution space of (PDE) (1.1) and (COV) (1.2), known from conformal field theory. However, the functions constructed there were not shown to be positive. As a by-product of Theorem 1.1, we establish positivity for these functions when $\kappa \in (0, 4) \setminus \mathbb{Q}$ thus identifying them with our functions of Theorem 1.1.

4.2 Total Partition Functions

In this section, we collect some results concerning the total partition functions

$$\mathcal{Z}^{(N)} := \sum_{\alpha \in \text{LP}_N} \mathcal{Z}_\alpha, \quad (4.5)$$

where $\{\mathcal{Z}_\alpha: \alpha \in \text{LP}\}$ is the collection of functions of Theorem 1.1. These functions were also called symmetric partition functions in [KP16], due to the property (4.6) below. In the range $\kappa \in (0, 4]$, the functions $\mathcal{Z}^{(N)}$ have explicit formulas for $\kappa = 2, 3$, and 4, given in Lemmas 4.9, 4.10 and 4.11.

Lemma 4.7. *The collection $\{\mathcal{Z}^{(N)}: N \geq 0\}$ of functions $\mathcal{Z}^{(N)}: \mathfrak{X}_{2N} \rightarrow \mathbb{R}_{>0}$ satisfies $\mathcal{Z}^{(N)} \in \mathcal{S}_N$ and $\mathcal{Z}^{(0)} = 1$, and the following asymptotics property: for all $j \in \{1, \dots, 2N-1\}$ and $\xi \in (x_{j-1}, x_{j+2})$,*

$$\lim_{x_j, x_{j+1} \rightarrow \xi} \frac{\mathcal{Z}^{(N)}(x_1, \dots, x_{2N})}{(x_{j+1} - x_j)^{-2h}} = \mathcal{Z}^{(N-1)}(x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N}). \quad (4.6)$$

Proof. The normalization $\mathcal{Z}^{(0)} = 1$ is clear, and we have $\mathcal{Z}^{(N)} \in \mathcal{S}_N$ by Proposition 4.6. The asymptotics (4.6) follow from the asymptotics (ASY) (1.3) of the pure partition functions \mathcal{Z}_α . \square

Corollary 4.8. *Let $\{F^{(N)}: N \geq 0\}$ be a collection of functions $F^{(N)} \in \mathcal{S}_N$ satisfying the asymptotics property (4.6) with the normalization $F^{(0)} = 1$. Then we have $F^{(N)} = \mathcal{Z}^{(N)}$ for all $N \geq 0$.*

Proof. After replacing (ASY) (1.3) by (4.6), the proof of Lemma 4.5 applies verbatim to show that the collection $\{F^{(N)}: N \geq 0\}$ is unique. Lemma 4.7 then shows that we have $F^{(N)} = \mathcal{Z}^{(N)}$ for all $N \geq 0$. \square

Next we give algebraic formulas of the total partition functions for $\kappa = 2, 3$ and 4. To state them for $\kappa = 2, 3$, we use the following notation. Let Π_N be the set of all partitions $\varpi = \{\{a_1, b_1\}, \dots, \{a_N, b_N\}\}$ of $\{1, \dots, 2N\}$ into N disjoint two-element subsets $\{a, b\} \subset \{1, \dots, 2N\}$, with the convention that $a_1 < a_2 < \dots < a_N$ and $a_j < b_j$ for all $j \in \{1, \dots, N\}$. Denote by $\text{sgn}(\varpi)$ the sign of the partition ϖ defined as the sign of the product $\prod (a - c)(a - d)(b - c)(b - d)$ over pairs of distinct elements $\{a, b\}, \{c, d\} \in \varpi$.

Lemma 4.9. *Let $\kappa = 2$. For all $N \geq 1$, we have*

$$\mathcal{Z}_{\text{LERW}}^{(N)}(x_1, \dots, x_{2N}) = \sum_{\varpi \in \Pi_N} \text{sgn}(\varpi) \det \left(\frac{1}{(x_{b_j} - x_{a_i})^2} \right)_{i,j=1}^N. \quad (4.7)$$

Proof. Consider the function $\tilde{\mathcal{Z}}_{\text{LERW}}^{(N)} := \sum \text{sgn}(\varpi) \det((x_{b_j} - x_{a_i})^{-2})$ on the right hand side. By [KKP17, Lemmas 4.4 and 4.5] and linearity, the function $\tilde{\mathcal{Z}}_{\text{LERW}}^{(N)}$ satisfies (PDE) (1.1) and (COV) (1.2) with $\kappa = 2$. It also clearly satisfies the bound (2.7). Also, if $N = 0$, then we have $\tilde{\mathcal{Z}}_{\text{LERW}}^{(0)} = 1$. Thus, by Corollary 4.8, it suffices to show that $\tilde{\mathcal{Z}}_{\text{LERW}}^{(N)}$ also satisfies the asymptotics property (4.6) with $\kappa = 2$. To prove this, fix $j \in \{1, \dots, 2N - 1\}$ and $\xi \in (x_{j-1}, x_{j+2})$. The limit in (4.6) with $\kappa = 2$ reads

$$\begin{aligned}
& \lim_{x_j, x_{j+1} \rightarrow \xi} (x_{j+1} - x_j)^2 \tilde{\mathcal{Z}}_{\text{LERW}}^{(N)}(x_1, \dots, x_{2N}) \\
&= \lim_{x_j, x_{j+1} \rightarrow \xi} (x_{j+1} - x_j)^2 \sum_{\varpi \in \Pi_N} \text{sgn}(\varpi) \det \left(\frac{1}{(x_{b_k} - x_{a_l})^2} \right)_{l,k=1}^N \\
&= \lim_{x_j, x_{j+1} \rightarrow \xi} (x_{j+1} - x_j)^2 \sum_{\varpi \in \Pi_N} \text{sgn}(\varpi) \sum_{\sigma \in \mathfrak{S}_N} \text{sgn}(\sigma) \prod_{k=1}^N \frac{1}{(x_{b_k} - x_{a_{\sigma(k)}})^2} \\
&= \lim_{x_j, x_{j+1} \rightarrow \xi} (x_{j+1} - x_j)^2 \sum_{\sigma \in \mathfrak{S}_N} \text{sgn}(\sigma) \sum_{\varpi \in \Pi_N} \text{sgn}(\varpi) \prod_{k=1}^N \frac{1}{(x_{b_k} - x_{a_{\sigma(k)}})^2}, \tag{4.8}
\end{aligned}$$

where \mathfrak{S}_N denotes the group of permutations of $\{1, \dots, N\}$. To evaluate this limit, for any pair of indices $k, l \in \{1, 2, \dots, N\}$, with $k \neq l$ we define the bijection

$$\begin{aligned}
\varphi_{l,k}: \{ \varpi \in \Pi_N \mid j = b_k \text{ and } j+1 = a_l \text{ in } \varpi \} &\longrightarrow \{ \varpi \in \Pi_N \mid j = a_l \text{ and } j+1 = b_k \text{ in } \varpi \} \\
\varphi_{l,k}(\varpi) &:= \left(\varpi \setminus \{ \{j', j\}, \{j+1, (j+1)'\} \} \right) \cup \{ \{j', j+1\}, \{j, (j+1)'\} \},
\end{aligned}$$

where j' and $(j+1)'$ denote the pairs of j and $j+1$ in ϖ , respectively. We have $\text{sgn}(\varphi_{l,k}(\varpi)) = -\text{sgn}(\varpi)$.

Consider a term in (4.8) with fixed $\sigma \in \mathfrak{S}_N$. Only terms where in $\varpi = \{ \{a_1, b_1\}, \dots, \{a_N, b_N\} \}$ we have for some $k \in \{1, 2, \dots, N\}$ either $j = a_{\sigma(k)}$ and $j+1 = b_k$, or $j = b_k$ and $j+1 = a_{\sigma(k)}$, can have a non-zero limit. With the bijections $\varphi_{\sigma(k),k}$, we may cancel all terms for which $\sigma(k) \neq k$. Thus, we are left with the terms for which $\{j, j+1\} = \{a_k, b_k\} \in \varpi$ and $\sigma(k) = k$, which allows us to reduce the sums over $\sigma \in \mathfrak{S}_N$ and $\varpi = \{ \{a_1, b_1\}, \dots, \{a_N, b_N\} \} \in \Pi_N$ into sums over $\tau \in \mathfrak{S}_{N-1}$ and $\hat{\varpi} = \{ \{c_1, d_1\}, \dots, \{c_{N-1}, d_{N-1}\} \} \in \Pi_{N-1}$, to obtain the asserted asymptotics property (4.6) with $\kappa = 2$:

$$\begin{aligned}
& \lim_{x_j, x_{j+1} \rightarrow \xi} (x_{j+1} - x_j)^2 \tilde{\mathcal{Z}}_{\text{LERW}}^{(N)}(x_1, \dots, x_{2N}) \\
&= \sum_{\varpi: \{j, j+1\} \in \varpi} \text{sgn}(\varpi) \sum_{\tau \in \mathfrak{S}_{N-1}} \text{sgn}(\tau) \lim_{x_j, x_{j+1} \rightarrow \xi} (x_{j+1} - x_j)^2 \prod_{\substack{1 \leq k \leq N, \\ b_k \neq j+1}} \frac{1}{(x_{b_k} - x_{a_{\tau(k)}})^2} \\
&= \sum_{\hat{\varpi} \in \Pi_{N-1}} \text{sgn}(\hat{\varpi}) \sum_{\tau \in \mathfrak{S}_{N-1}} \text{sgn}(\tau) \prod_{k=1}^{N-1} \frac{1}{(x_{d_k} - x_{c_{\tau(k)}})^2} \\
&= \sum_{\hat{\varpi} \in \Pi_{N-1}} \text{sgn}(\hat{\varpi}) \det \left(\frac{1}{(x_{c_k} - x_{d_l})^2} \right)_{k,l=1}^{N-1} \\
&= \tilde{\mathcal{Z}}_{\text{LERW}}^{(N-1)}(x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N}).
\end{aligned}$$

This concludes the proof. \square

Lemma 4.10. *Let $\kappa = 3$. For all $N \geq 1$, we have*

$$\mathcal{Z}_{\text{Ising}}^{(N)}(x_1, \dots, x_{2N}) = \text{pf} \left(\frac{1}{x_{b_j} - x_{a_i}} \right)_{i,j=1}^N = \sum_{\varpi \in \Pi_N} \text{sgn}(\varpi) \left(\prod_{\{a,b\} \in \varpi} \frac{1}{x_b - x_a} \right). \tag{4.9}$$

Proof. It was proved in [KP16, Proposition 4.6] that the function $\tilde{\mathcal{Z}}_{\text{Ising}}^{(N)} := \sum \text{sgn}(\varpi) \left(\prod \frac{1}{x_b - x_a} \right)$, on the right hand side satisfies (PDE) (1.1) and (COV) (1.2) with $\kappa = 3$, and that it also has the asymptotics property (4.6) with $\kappa = 3$. Moreover, this function obviously satisfies the bound (2.7), and if $N = 0$, then we have $\tilde{\mathcal{Z}}_{\text{Ising}}^{(0)} = 1$. The claim then follows from Corollary 4.8. \square

Lemma 4.11. *Let $\kappa = 4$. For all $N \geq 1$, we have*

$$\mathcal{Z}_{\text{GFF}}^{(N)}(x_1, \dots, x_{2N}) = \prod_{1 \leq k < l \leq 2N} (x_l - x_k)^{\frac{1}{2}(-1)^{l-k}}. \quad (4.10)$$

Proof. It was proved in [KP16, Proposition 4.8] that the function $\tilde{\mathcal{Z}}_{\text{GFF}}^{(N)} := \prod (x_l - x_k)^{\frac{1}{2}(-1)^{l-k}}$ on the right hand side satisfies (PDE) (1.1) and (COV) (1.2) with $\kappa = 4$, and that it also has the asymptotics property (4.6) with $\kappa = 4$. Moreover, this function obviously satisfies the bound (2.7), and if $N = 0$, then we have $\tilde{\mathcal{Z}}_{\text{GFF}}^{(0)} = 1$. The claim then follows from Corollary 4.8. \square

To finish, we record a general asymptotics property for $\mathcal{Z}_{\text{GFF}}^{(N)}$ that will be needed in Section 5.

Corollary 4.12. *Let $\kappa = 4$. For all $k \in \{1, \dots, N\}$ and $\xi < x_{2k+1} < \dots < x_{2N}$, we have*

$$\lim_{\substack{\tilde{x}_1, \dots, \tilde{x}_{2k} \rightarrow \xi, \\ \tilde{x}_i \rightarrow x_i \text{ for } 2k < i \leq 2N}} \frac{\mathcal{Z}_{\text{GFF}}^{(N)}(\tilde{x}_1, \dots, \tilde{x}_{2N})}{\mathcal{Z}_{\text{GFF}}^{(k)}(\tilde{x}_1, \dots, \tilde{x}_{2k})} = \mathcal{Z}_{\text{GFF}}^{(N-k)}(x_{2k+1}, \dots, x_{2N}).$$

Proof. This follows from the explicit formula (4.10) for $\mathcal{Z}_{\text{GFF}}^{(N)}$. \square

4.3 Global Multiple SLEs are Local Multiple SLEs

In this section, we show that the global SLE_κ probability measures $\mathbb{Q}_\alpha^\#$ constructed in Section 3.1 agree with another natural definition of multiple SLEs — the *local* N - SLE_κ . The latter measures are defined in terms of their Loewner chain description, which allows one to treat the random curves as growth processes. We first recall the definition of a local multiple SLE_κ from [Dub07] and [KP16, Appendix A].

Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and fix $2N$ boundary points x_1, \dots, x_{2N} along $\partial\Omega$ in counterclockwise order. The localization neighborhoods U_1, \dots, U_{2N} are assumed to be closed subsets of $\bar{\Omega}$ such that $\Omega \setminus U_j$ are simply connected and $U_j \cap U_k = \emptyset$ for $j \neq k$. The local N - SLE_κ in Ω , started from (x_1, \dots, x_{2N}) and localized in (U_1, \dots, U_{2N}) , is a probability measure on $2N$ -tuples of oriented unparameterized curves $(\gamma_1, \dots, \gamma_{2N})$. For convenience, we choose a parameterization of the curves by $t \in [0, 1]$, so that for each j , the curve $\gamma_j : [0, 1] \rightarrow U_j$ starts at $\gamma_j(0) = x_j$ and ends at $\gamma_j(1) \in \partial(\Omega \setminus U_j)$. The local N - SLE_κ is an indexed collection of probability measures on $(\gamma_1, \dots, \gamma_{2N})$:

$$\mathbf{P} = \left(\mathbf{P}_{(U_1, \dots, U_{2N})}^{(\Omega; x_1, \dots, x_{2N})} \right)_{\Omega; x_1, \dots, x_{2N}; U_1, \dots, U_{2N}}$$

This collection is required to satisfy conformal invariance (CI), domain Markov property (DMP), and absolute continuity of marginals with respect to the chordal SLE_κ (MARG):

(CI) If $(\gamma_1, \dots, \gamma_{2N}) \sim \mathbf{P}_{(U_1, \dots, U_{2N})}^{(\Omega; x_1, \dots, x_{2N})}$ and $\varphi: \Omega \rightarrow \tilde{\Omega}$ is a conformal map, then

$$(\varphi(\gamma_1), \dots, \varphi(\gamma_{2N})) \sim \mathbf{P}_{(\varphi(U_1), \dots, \varphi(U_{2N}))}^{(\tilde{\Omega}; \varphi(x_1), \dots, \varphi(x_{2N}))}.$$

(DMP) Let τ_j be stopping times for γ_j , for $j \in \{1, \dots, N\}$. Given initial segments $(\gamma_1[0, \tau_1], \dots, \gamma_{2N}[0, \tau_{2N}])$, the conditional law of the remaining parts $(\gamma_1|_{[\tau_1, 1]}, \dots, \gamma_{2N}|_{[\tau_{2N}, 1]})$ is $\mathbf{P}_{(\tilde{U}_1, \dots, \tilde{U}_{2N})}^{(\tilde{\Omega}; \tilde{x}_1, \dots, \tilde{x}_{2N})}$, where $\tilde{\Omega}$ is the component of $\Omega \setminus \bigcup_{j=1}^{2N} \gamma_j[0, \tau_j]$ containing all tips $\tilde{x}_j = \gamma_j(\tau_j)$ on its boundary and $\tilde{U}_j = U_j \cap \tilde{\Omega}$.

(MARG) There exist smooth functions $F_j: \mathfrak{X}_{2N} \rightarrow \mathbb{R}$, for $j \in \{1, \dots, 2N\}$, such that for the domain $\Omega = \mathbb{H}$, boundary points $x_1 < \dots < x_{2N}$, and their localization neighborhoods U_1, \dots, U_{2N} , the marginal law of γ_j under $\mathbf{P}_{(U_1, \dots, U_{2N})}^{(\mathbb{H}; x_1, \dots, x_{2N})}$ is the Loewner evolution driven by W_t which is the solution to

$$\begin{aligned} dW_t &= \sqrt{\kappa} dB_t + F_j(V_t^1, \dots, V_t^{j-1}, W_t, V_t^{j+1}, \dots, V_t^{2N}) dt, & W_0 &= x_j \\ dV_t^i &= \frac{2dt}{V_t^i - W_t}, & V_0^i &= x_i, \quad \text{for } i \neq j. \end{aligned} \quad (4.11)$$

Remark 4.13. *It follows from the definition that the local N -SLE $_{\kappa}$ is consistent under restriction to smaller localization neighborhoods, see [KP16, Proposition A.2].*

J. Dubédat proved in [Dub07] that the local N -SLE $_{\kappa}$ processes are classified by partition functions \mathcal{Z} as described in the next proposition. The convex structure of the set of the local N -SLE $_{\kappa}$ was further studied by K. Kytölä and E. Peltola in [KP16].

Proposition 4.14. *Fix $\kappa \in (0, 8)$.*

- *Suppose \mathbf{P} is a local N -SLE $_{\kappa}$. Then there exists a function $\mathcal{Z}: \mathfrak{X}_{2N} \rightarrow \mathbb{R}_{>0}$ satisfying (PDE) (1.1) and (COV) (1.2), such that for all $j \in \{1, \dots, 2N\}$, the drift functions in (MARG) take the form $F_j = \kappa(\partial_j \mathcal{Z})/\mathcal{Z}$. Such a function \mathcal{Z} is determined up to a multiplicative constant.*
- *Suppose $\mathcal{Z}: \mathfrak{X}_{2N} \rightarrow \mathbb{R}_{>0}$ satisfies (PDE) (1.1) and (COV) (1.2). Then the random collection of curves obtained by the Loewner chain in (MARG) with $F_j = \kappa(\partial_j \mathcal{Z})/\mathcal{Z}$ for all $j \in \{1, \dots, 2N\}$ is a local N -SLE $_{\kappa}$. Two functions \mathcal{Z} and $\tilde{\mathcal{Z}}$ give rise to the same local N -SLE $_{\kappa}$ if and only if $\mathcal{Z} = \text{const.} \times \tilde{\mathcal{Z}}$.*
- *Convex combinations of the local N -SLE $_{\kappa}$ correspond to positive linear combinations of their partition functions (modulo multiplicative constants), as detailed in [KP16, Theorem A.4(c)].*

Proof. This follows from results in [Dub07, Gra07, Kyt07] and [KP16, Theorem A.4]. \square

For each (normalized) partition function $\mathcal{Z}: \mathfrak{X}_{2N} \rightarrow \mathbb{R}_{>0}$, that is, a solution to (PDE) (1.1) and (COV) (1.2), we call the collection \mathbf{P} of probability measures for which we have $F_j = \kappa(\partial_j \mathcal{Z})/\mathcal{Z}$ for all $j \in \{1, \dots, 2N\}$ in (MARG) *the local N -SLE $_{\kappa}$ with partition function \mathcal{Z}* . Next we prove that our construction of the global N -SLE $_{\kappa}$ measures in the previous section is consistent with this local definition.

Lemma 4.15. *Fix $\kappa \in (0, 4]$. Any global N -SLE $_{\kappa}$ satisfying (MARG) is a local N -SLE $_{\kappa}$ when it is restricted to any localization neighborhoods. For any $\alpha \in \text{LP}_N$, the restriction of the global N -SLE $_{\kappa}$ probability measure $\mathbb{Q}_{\alpha}^{\#}$ associated to α (constructed in Proposition 3.3) to any localization neighborhoods is the same as the local N -SLE $_{\kappa}$ with partition function \mathcal{Z}_{α} given by (3.7).*

Proof. Fix $\Omega = \mathbb{H}$, boundary points $x_1 < \dots < x_{2N}$, localization neighborhoods (U_1, \dots, U_{2N}) , and a link pattern $\alpha = \{\{a_1, b_1\}, \dots, \{a_N, b_N\}\} \in \text{LP}_N$. Suppose that (η_1, \dots, η_N) is a global N -SLE $_{\kappa}$ associated to α . Then for $j \in \{1, \dots, N\}$, the curve η_j connects x_{a_j} to x_{b_j} . Denote by $\hat{\eta}_j$ the time-reversal of η_j . Let τ_j be the first time when η_j exits U_{a_j} , and define γ_{a_j} to be the curve $(\eta_j(t) : 0 \leq t \leq \tau_j)$. Let $\hat{\tau}_j$ be the first time that $\hat{\eta}_j$ exits U_{b_j} , and define γ_{b_j} to be the curve $(\hat{\eta}_j(t) : 0 \leq t \leq \hat{\tau}_j)$. By conformal invariance of the SLE $_{\kappa}$, the law of the collection $(\gamma_1, \dots, \gamma_{2N})$ satisfies (CI), and it also satisfies (DMP) thanks to the domain Markov property and reversibility of the SLE $_{\kappa}$. Therefore, any global N -SLE $_{\kappa}$ satisfying (MARG) is a local N -SLE $_{\kappa}$ when it is restricted to any localization neighborhoods.

Suppose then $(\eta_1, \dots, \eta_N) \sim \mathbb{Q}_{\alpha}^{\#}(\mathbb{H}; x_1, \dots, x_{2N})$ and define $(\gamma_1, \dots, \gamma_{2N})$ as above. We only need to check the property (MARG). Without loss of generality, we show it for γ_1 . From the proof of Lemma 4.4,

we see that the marginal law of γ_1 under $\mathbb{Q}_\alpha^\#$ is absolutely continuous with respect to the SLE_κ in \mathbb{H} from x_1 to ∞ , and the Radon-Nikodym derivative M_t/M_0 is given by the local martingale

$$M_t := \prod_{j=2}^{2N} g'_t(x_j)^h \times \mathcal{Z}_\alpha(W_t, g_t(x_2), \dots, g_t(x_{2N})).$$

This implies that the curve γ_1 has the same driving function as in (MARG) for $j = 1$, with drift function $F_1 = \kappa(\partial_1 \mathcal{Z}_\alpha)/\mathcal{Z}_\alpha$. Because, by Lemma 4.4, \mathcal{Z}_α is smooth, F_1 is smooth. This completes the proof. \square

We finish this section with the proof of Corollary 1.2.

Corollary 1.2. *Let $\kappa \in (0, 4]$. For any $\alpha \in \text{LP}_N$, there exists a local $N\text{-SLE}_\kappa$ with partition function \mathcal{Z}_α . The functions $\{\mathcal{Z}_\alpha : \alpha \in \text{LP}\}$ are linearly independent. For any $N \geq 1$, the convex hull of the local $N\text{-SLE}_\kappa$ corresponding to \mathcal{Z}_α for $\alpha \in \text{LP}_N$ has dimension $C_N - 1$. The C_N local $N\text{-SLE}_\kappa$ probability measures with pure partition functions \mathcal{Z}_α are the extremal points of this convex set.*

Proof. Theorem 1.1 and Proposition 4.14 show that local multiple SLEs exist for any \mathcal{Z}_α , $\alpha \in \text{LP}_N$. Linear independence of the functions \mathcal{Z}_α was proved in Proposition 4.6. With the classification of local multiple SLEs from Proposition 4.14, the last assertion then follows from Theorem 1.1. \square

5 Gaussian Free Field

This section is devoted to the study of the level lines of the Gaussian free field (GFF) with alternating boundary conditions, generalizing the Dobrushin boundary conditions $-\lambda, +\lambda$ on two complementary boundary segments to $-\lambda, +\lambda, \dots, -\lambda, +\lambda$ on $2N$ boundary segments. Much of these level lines is already known: a level line starting from a boundary point is an $\text{SLE}_4(\rho)$ process, and the level lines can be coupled with the GFF in such a way that they are almost surely determined by the field [Dub09, SS13, MS16a].

We are interested in the probabilities that the level lines form a particular connectivity pattern, encoded in $\alpha \in \text{LP}_N$. The main result of this section, Theorem 1.4, states that this probability is given by the pure partition functions \mathcal{Z}_α for multiple SLE_κ with $\kappa = 4$. We prove Theorem 1.4 in Section 5.5. In Section 6, we find explicit formulas for these connection probabilities, see Equation (1.8) in Theorem 1.5.

5.1 SLE with Force Points

$\text{SLE}_\kappa(\rho)$ processes are variants of the SLE_κ where one keeps track of additional points on the boundary. Let $\underline{y}^L = (y^{1,L} < \dots < y^{l,L} \leq 0)$ and $\underline{y}^R = (0 \leq y^{1,R} < \dots < y^{r,R})$ and $\underline{\rho}^L = (\rho^{1,L}, \dots, \rho^{l,L})$ and $\underline{\rho}^R = (\rho^{1,R}, \dots, \rho^{r,R})$, where $\rho^{i,q} \in \mathbb{R}$ for $q \in \{L, R\}$ and $i \in \mathbb{N}$. An $\text{SLE}_\kappa(\underline{\rho}^L; \underline{\rho}^R)$ process with force points $(\underline{y}^L; \underline{y}^R)$ is the Loewner evolution driven by W_t that solves the following system of integrated SDEs:

$$\begin{aligned} W_t &= \sqrt{\kappa} B_t + \sum_{i=1}^l \int_0^t \frac{\rho^{i,L} ds}{W_s - V_s^{i,L}} + \sum_{i=1}^r \int_0^t \frac{\rho^{i,R} ds}{W_s - V_s^{i,R}}, \\ V_t^{i,q} &= y^{i,q} + \int_0^t \frac{2ds}{V_s^{i,q} - W_s}, \quad \text{for } q \in \{L, R\} \text{ and } i \in \mathbb{N}, \end{aligned} \tag{5.1}$$

where B_t is an one-dimensional Brownian motion. Note that the process $V_t^{i,q}$ is the evolution of the point $y^{i,q}$, and we may write $g_t(y^{i,q})$ for $V_t^{i,q}$. We define the *continuation threshold* of the $\text{SLE}_\kappa(\underline{\rho}^L; \underline{\rho}^R)$ to be the infimum of the time t for which

$$\text{either } \sum_{i: V_t^{i,L}=W_t} \rho^{i,L} \leq -2, \quad \text{or} \quad \sum_{i: V_t^{i,R}=W_t} \rho^{i,R} \leq -2.$$

By [MS16a], the $\text{SLE}_\kappa(\underline{\rho}^L; \underline{\rho}^R)$ process is well-defined up to the continuation threshold, and it is almost surely generated by a continuous curve up to and including the continuation threshold.

5.2 Level Lines of the GFF

Next, we introduce the Gaussian Free Field and its level lines, and summarize some of their properties that will be useful later. We refer to the literature for details: see e.g. [She07, SS13, MS16a, WW17].

Suppose that $D \subsetneq \mathbb{C}$ is a simply connected domain with non-polar boundary. For two functions $f, g \in L^2(D)$, we denote by (f, g) their inner product in $L^2(D)$, that is, $(f, g) := \int_D f(z)g(z)d^2z$, where d^2z is the Lebesgue area measure. We denote by $H_s(D)$ the space of real-valued smooth functions which are compactly supported in D . This space has a *Dirichlet inner product* defined by

$$(f, g)_\nabla := \frac{1}{2\pi} \int_D \nabla f(z) \cdot \nabla g(z) d^2z.$$

We denote by $H(D)$ the Hilbert space completion of $H_s(D)$ with respect to the Dirichlet inner product.

The *zero-boundary* GFF on D is a random sum of the form $h = \sum_{j=1}^\infty \alpha_j f_j$, where α_j are i.i.d. standard normal random variables and $(f_j)_{j \geq 0}$ an orthonormal basis for $H(D)$. This sum almost surely diverges within $H(D)$; however, it does converge almost surely in the space of distributions — that is, as $n \rightarrow \infty$, the limit of $\sum_{j=1}^n \alpha_j (f_j, g)$ exists almost surely for all $g \in H_s(D)$ and we may define $(h, g) := \sum_{j=1}^\infty \alpha_j (f_j, g)$. The limiting value as a function of g is almost surely a continuous functional on $H_s(D)$. In general, for any harmonic function h_0 on D , we define the GFF *with boundary data* h_0 by $h := \tilde{h} + h_0$ where \tilde{h} is a zero-boundary GFF in D .

We next introduce the level lines of GFF and list some of their properties proved in [SS13, SS13, MS16a, WW17]. Let $K = (K_t, t \geq 0)$ be an $\text{SLE}_4(\rho^L; \rho^R)$ process with force points $(\underline{y}^L; \underline{y}^R)$ where $W, V^{i,q}$ solve the SDEs (5.1). Let $(g_t, t \geq 0)$ be the corresponding family of conformal maps and set $f_t := g_t - W_t$. Let h_t^0 be the harmonic function on \mathbb{H} with boundary values given by

$$\begin{cases} -\lambda(1 + \sum_{i=0}^j \rho^{i,L}), & \text{if } x \in (f_t(y^{j+1,L}), f_t(y^{j,L})), \\ +\lambda(1 + \sum_{i=0}^j \rho^{i,R}), & \text{if } x \in (f_t(y^{j,R}), f_t(y^{j+1,R})), \end{cases}$$

where $\lambda = \pi/2$ and $\rho^{0,L} = \rho^{0,R} = 0, y^{0,L} = 0_-, y^{l+1,L} = -\infty, y^{0,R} = 0_+, y^{r+1,R} = \infty$ by convention. Define $h_t(z) := h_t^0(f_t(z))$. By [Dub09, SS13, MS16a], there exists a coupling (h, K) where $h = \tilde{h} + h_0$, with \tilde{h} a zero-boundary GFF in \mathbb{H} , such that the following is true. Suppose that τ is any K -stopping time before the continuation threshold. Then the conditional law of h restricted to $\mathbb{H} \setminus K_\tau$ given K_τ is the same as the law of $h_\tau + \tilde{h} \circ f_\tau$. Furthermore, in this coupling, the process K is almost surely determined by h . We refer to the $\text{SLE}_4(\rho^L; \rho^R)$ in this coupling as the *level line* of the field h . In particular, if the boundary value of h is $-\lambda$ on \mathbb{R}_- and λ on \mathbb{R}_+ , then the level line of h starting from 0 has the law of the chordal SLE_4 from 0 to ∞ . In this case, we say that the field has *Dobrushin boundary conditions*. In general, for $u \in \mathbb{R}$, the level line of h with height u is the level line of $h - u$.

Suppose h is a GFF in \mathbb{H} with piecewise constant boundary values and let η be the level line of h starting from 0. For $0 < x < y$, assume that the boundary value of h is a constant c on (x, y) . Consider the intersection of η with the interval $[x, y]$. The following facts were proved in [WW17, Section 2.5]. First, if $|c| \geq \lambda$, then $\eta \cap (x, y) = \emptyset$ almost surely; second, if $c \geq \lambda$, then η can never hit the point x ; third, if $c \leq -\lambda$, then η can never hit the point y , but it may hit the point x , and when it hits x , it meets its continuation threshold and cannot continue. In this case, we say that η *terminates at* x .

5.3 Pair of Level Lines

Fix four points $x_1 < x_2 < x_3 < x_4$ on the real line and suppose that h is the GFF in \mathbb{H} with the following boundary values (see also Figure 5.1):

$$-\lambda \text{ on } (-\infty, x_1), \quad +\lambda \text{ on } (x_1, x_2), \quad -\lambda \text{ on } (x_2, x_3), \quad +\lambda \text{ on } (x_3, x_4), \quad -\lambda \text{ on } (x_4, \infty).$$

Let η_1 (resp. η_2) be the level line of h starting from x_1 (resp. x_3). The two curves η_1 and η_2 cannot hit each other, and there are two cases for the possible end points of η_1 and η_2 , illustrated in Figure 5.1:

Case $\overbrace{\quad\quad}^{\quad}$, where η_1 terminates at x_4 and η_2 terminates at x_2 ; and Case $\underbrace{\quad\quad}_{\quad}$, where η_1 terminates at x_2 and η_2 terminates at x_4 . Both cases have a positive chance. As a warm-up, we calculate the probabilities for these two cases in Lemma 5.3. Note that, given η_1 , the curve η_2 is the level line of the field in $\mathbb{H} \setminus \eta_1$ which has Dobrushin boundary conditions. Therefore, in either case, the conditional law of η_2 given η_1 is the chordal SLE₄ and, similarly, the conditional law of η_1 given η_2 is the chordal SLE₄.

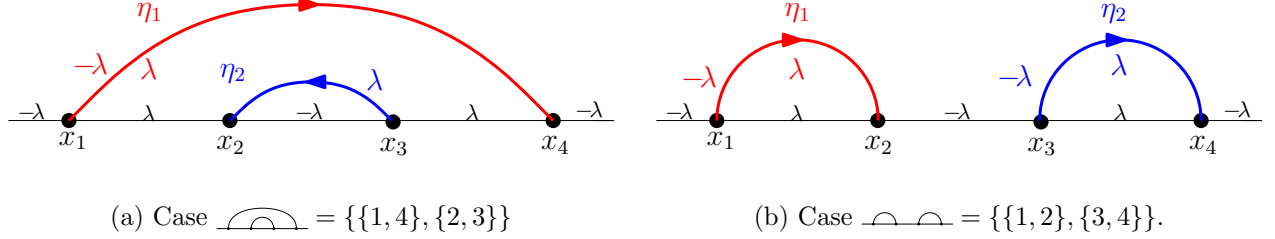


Figure 5.1: Two level lines of the GFF. The conditional law of η_1 given η_2 is the chordal SLE₄ and the conditional law of η_2 given η_1 is the chordal SLE₄.

Remark 5.1. The following trivial fact will be important later: For $x_1 < x_2 < x_3 < x_4$, we have

$$0 \leq \frac{(x_4 - x_1)(x_3 - x_2)}{(x_4 - x_2)(x_3 - x_1)} \leq 1.$$

We will also make use of the following technical observation.

Lemma 5.2. Fix $x_1 < x_2 < x_3 < x_4$. Suppose η is a continuous simple curve in \mathbb{H} starting from x_1 and terminating at x_4 at time T , which hits \mathbb{R} only at $\{x_1, x_4\}$. Let $(W_t, 0 \leq t \leq T)$ be its Loewner driving function and $(g_t, 0 \leq t \leq T)$ the corresponding conformal maps. Define, for $t < T$,

$$\Delta_t = \frac{(g_t(x_4) - W_t)(g_t(x_3) - g_t(x_2))}{(g_t(x_4) - g_t(x_2))(g_t(x_3) - W_t)}.$$

Then we have $0 \leq \Delta_t \leq 1$ for all $t < T$, and $\Delta_t \rightarrow 0$ as $t \rightarrow T$.

Proof. The bound $0 \leq \Delta_t \leq 1$ follows from Remark 5.1 and the fact that $W_t < g_t(x_2) < g_t(x_3) < g_t(x_4)$. It remains to check the limit of Δ_t as $t \rightarrow T$. To simplify notations, we denote $g_t(x_2) - W_t$ by X_{21} and $g_t(x_3) - g_t(x_2)$ by X_{32} , and $g_t(x_4) - g_t(x_3)$ by X_{43} . Then we have

$$\Delta_t = \frac{(X_{43} + X_{32} + X_{21})X_{32}}{(X_{43} + X_{32})(X_{32} + X_{21})} = \frac{X_{32}/X_{21} + X_{32}/X_{43} + X_{32}^2/(X_{21}X_{43})}{1 + X_{32}/X_{21} + X_{32}/X_{43} + X_{32}^2/(X_{21}X_{43})}.$$

To show that $\Delta_t \rightarrow 0$ as $t \rightarrow T$, it suffices to show that

$$X_{32}/X_{21} \rightarrow 0, \quad \text{and} \quad X_{32}/X_{43} \rightarrow 0. \tag{5.2}$$

For $z \in \mathbb{C}$, denote by \mathbb{P}^z the law of Brownian motion in \mathbb{C} started from z . Let τ be the first time when B exits $\mathbb{H} \setminus \eta[0, t]$. Then by [Law05, Remark 3.50], we have

$$X_{43} = \lim_{y \rightarrow \infty} y \mathbb{P}^{yi}[B_\tau \in (x_3, x_4)], \quad X_{32} = \lim_{y \rightarrow \infty} y \mathbb{P}^{yi}[B_\tau \in (x_2, x_3)],$$

and X_{21} is the same limit of the probability that B_τ belongs the union of the right side of $\eta[0, t]$ and (x_1, x_2) . Property (5.2) follows from this. \square

Lemma 5.3. Set $\overbrace{\quad\quad}^{\quad} = \{\{1, 4\}, \{2, 3\}\}$ and $\underbrace{\quad\quad}_{\quad} = \{\{1, 2\}, \{3, 4\}\}$. Let $P_{\overbrace{\quad\quad}^{\quad}}$ be the probability for Case $\overbrace{\quad\quad}^{\quad}$ and $P_{\underbrace{\quad\quad}_{\quad}}$ the probability for Case $\underbrace{\quad\quad}_{\quad}$, as in Figure 5.1. Then we have

$$P_{\overbrace{\quad\quad}^{\quad}} = \frac{(x_4 - x_3)(x_2 - x_1)}{(x_4 - x_2)(x_3 - x_1)} \quad \text{and} \quad P_{\underbrace{\quad\quad}_{\quad}} = 1 - P_{\overbrace{\quad\quad}^{\quad}} = \frac{(x_4 - x_1)(x_3 - x_2)}{(x_4 - x_2)(x_3 - x_1)}.$$

Proof. We know that $\eta := \eta_1$ is an $\text{SLE}_4(-2, +2, -2)$ process with force points (x_2, x_3, x_4) . If T is the continuation threshold of η , then Case $\overbrace{\quad\quad\quad}$ corresponds to $\{\eta(T) = x_4\}$ and Case $\underbrace{\quad\quad\quad}$ to $\{\eta(T) = x_2\}$. Define, for $t < T$,

$$M_t = \frac{(g_t(x_4) - W_t)(g_t(x_3) - g_t(x_2))}{(g_t(x_4) - g_t(x_2))(g_t(x_3) - W_t)}.$$

Using Itô's Formula, one can check that M_t is a local martingale, and it is bounded by Remark 5.1: we have $0 \leq M_t \leq 1$ for $t < T$. Moreover, by Lemma 5.2, we have almost surely, as $t \rightarrow T$,

$$M_t \rightarrow 1, \quad \text{when } \eta(t) \rightarrow x_2 \quad \text{and} \quad M_t \rightarrow 0, \quad \text{when } \eta(t) \rightarrow x_4.$$

Therefore, Optional Stopping Theorem implies

$$P_{\underbrace{\quad\quad\quad}} = \mathbb{P}[\eta(T) = x_2] = \mathbb{E}[M_T] = M_0 = \frac{(x_4 - x_1)(x_3 - x_2)}{(x_4 - x_2)(x_3 - x_1)}.$$

The formula for the probability $P_{\overbrace{\quad\quad\quad}}$ then follows by a direct calculation. \square

5.4 Connection Probabilities for the Level Lines

Fix $N \geq 2$ and $x_1 < x_2 < \dots < x_{2N}$. Suppose h is a GFF in \mathbb{H} with alternating boundary conditions:

$$-\lambda \text{ on } (x_{2j}, x_{2j+1}), \text{ for } j \in \{0, 1, \dots, N\} \quad \text{and} \quad +\lambda \text{ on } (x_{2j+1}, x_{2j+2}), \text{ for } j \in \{0, 1, \dots, N-1\},$$

with the convention that $x_0 = -\infty$ and $x_{2N+1} = \infty$. For $j \in \{1, \dots, N\}$, let η_j be the level line of h starting from x_{2j-1} . The possible terminal points of η_j are x_k 's with an even index k . The level lines η_1, \dots, η_N do not hit each other, so their end points form a (planar) link pattern $\mathcal{A} \in \text{LP}_N$. In Lemma 5.6, we derive the connection probability $P_\alpha := \mathbb{P}[\mathcal{A} = \alpha]$ for each $\alpha \in \text{LP}_N$. To this end, we use the next lemmas, which relate martingales for level lines with solutions of the system (PDE) (1.1) with $\kappa = 4$.

Lemma 5.4. *Let $\eta = \eta_1$ be the level line of h starting from x_1 , let W_t be its driving function, and $(g_t, t \geq 0)$ the corresponding family of conformal maps. Denote $X_{j1} := g_t(x_j) - W_t$ and $X_{ji} := g_t(x_j) - g_t(x_i)$ for $i, j \in \{2, 3, \dots, 2N\}$. For any subset $S \subset \{1, 2, \dots, 2N\}$ containing 1, define*

$$M_t^{(S)} = \prod_{1 \leq i < j \leq 2N} X_{ji}^{\delta(i,j)}, \quad \text{where } \delta(i,j) = \begin{cases} 0, & \text{if } i, j \in S, \text{ or } i, j \notin S, \\ (-1)^{1+i+j}, & \text{if } i \in S \text{ and } j \notin S, \text{ or } i \notin S \text{ and } j \in S. \end{cases}$$

Then $M_t^{(S)}$ is a local martingale.

Proof. The level line η is an $\text{SLE}_4(-2, +2, \dots, -2)$ process with force points (x_2, \dots, x_{2N}) . Recall from (5.1) that its driving function satisfies the SDE

$$dW_t = 2dB_t + \sum_{i=2}^{2N} \frac{-\rho_i dt}{g_t(x_i) - W_t}, \quad \text{where } \rho_i = 2(-1)^{i+1}. \quad (5.3)$$

Rewrite $M_t^{(S)}$ as follows:

$$M_t^{(S)} = \prod_{j=2}^{2N} X_{j1}^{\delta_j} \prod_{2 \leq i < j \leq 2N} X_{ji}^{\delta(i,j)}, \quad \text{where } \delta_j = \delta(1, j).$$

By Itô's Formula, we have

$$\begin{aligned} \frac{dM_t^{(S)}}{M_t^{(S)}} &= \sum_{j=2}^{2N} \frac{\delta_j}{X_{j1}} \left(\frac{2dt}{X_{j1}} - dW_t \right) + \sum_{2 \leq i < j \leq 2N} \frac{\delta(i, j)}{X_{ji}} \left(\frac{2dt}{X_{j1}} - \frac{2dt}{X_{i1}} \right) \\ &\quad + \sum_{j=2}^{2N} \frac{2\delta_j(\delta_j - 1)dt}{X_{j1}^2} + \sum_{2 \leq i < j \leq 2N} \frac{4\delta_i\delta_j dt}{X_{j1}X_{i1}} \\ &= \sum_{j=2}^{2N} \frac{2\delta_j^2 dt}{X_{j1}^2} + \sum_{j=2}^{2N} \sum_{i=2}^{2N} \frac{\delta_j \rho_i dt}{X_{j1}X_{i1}} + \sum_{2 \leq i < j \leq 2N} \left(\frac{-2\delta(i, j) + 4\delta_i\delta_j}{X_{j1}X_{i1}} \right) dt - \sum_{j=2}^{2N} \frac{2\delta_j dB_t}{X_{j1}}. \end{aligned}$$

For any S containing 1, one checks that the coefficient of the term dt/X_{j1}^2 for $j \in \{2, \dots, 2N\}$ is

$$2\delta_j^2 + \delta_j \rho_j = 0,$$

and the coefficient of the term $dt/(X_{j1}X_{i1})$ for $i, j \in \{2, \dots, 2N\}$, $i < j$, is

$$\delta_j \rho_i + \delta_i \rho_j - 2\delta(i, j) + 4\delta_i\delta_j = 0.$$

Therefore, $M_t^{(S)}$ is a local martingale. \square

Lemma 5.5. *Let $\eta = \eta_1$ be the level line of h starting from x_1 , let W_t be its driving function, and $(g_t, t \geq 0)$ the corresponding family of conformal maps. For a smooth function $\mathcal{U}: \mathfrak{X}_{2N} \rightarrow \mathbb{R}$, the ratio*

$$M_t(\mathcal{U}) := \frac{\mathcal{U}(W_t, g_t(x_2), \dots, g_t(x_{2N}))}{\mathcal{Z}_{\text{GFF}}^{(N)}(W_t, g_t(x_2), \dots, g_t(x_{2N}))}$$

is a local martingale if and only if \mathcal{U} satisfies (PDE) (1.1) with $i = 1$ and $\kappa = 4$.

Proof. Recall the SDE (5.3) for W_t . Lemma 4.11 gives an explicit formula for the function $\mathcal{Z} := \mathcal{Z}_{\text{GFF}}^{(N)}$. Using this, one verifies that \mathcal{Z} satisfies the following differential equation: for $\mathbf{x} = (x_1, \dots, x_{2N}) \in \mathfrak{X}_{2N}$,

$$\left(4\partial_1 + \sum_{j=2}^{2N} \frac{\rho_j}{x_j - x_1} \right) \mathcal{Z}(\mathbf{x}) = 0. \quad (5.4)$$

Furthermore, \mathcal{Z} satisfies (PDE) (1.1) with $i = 1$ and $\kappa = 4$:

$$\mathcal{D}^{(1)} \mathcal{Z}(\mathbf{x}) = 0, \quad \text{where } \mathcal{D}^{(1)} := 2\partial_1^2 + \sum_{j=2}^{2N} \left(\frac{2\partial_j}{x_j - x_1} - \frac{1}{2(x_j - x_1)^2} \right). \quad (5.5)$$

We denote $\mathbf{Y} := (W_t, g_t(x_2), \dots, g_t(x_{2N}))$, and $X_{j1} := g_t(x_j) - W_t$ and $X_{ji} := g_t(x_j) - g_t(x_i)$ for $i, j \in \{2, 3, \dots, 2N\}$. By Itô's Formula, any (regular enough) function $F(x_1, \dots, x_{2N})$ satisfies

$$\begin{aligned} dF(\mathbf{Y}) &= 2\partial_1 F(\mathbf{Y}) dB_t + \left(2\partial_1^2 + \sum_{j=2}^{2N} \left(\frac{2\partial_j}{X_{j1}} - \frac{\rho_j \partial_1}{X_{j1}} \right) \right) F(\mathbf{Y}) dt \\ &= 2\partial_1 F(\mathbf{Y}) dB_t + \left(\mathcal{D}^{(1)} + \sum_{j=2}^{2N} \left(\frac{1}{2X_{j1}^2} - \frac{\rho_j \partial_1}{X_{j1}} \right) \right) F(\mathbf{Y}) dt. \end{aligned}$$

Combining with (5.4) and (5.5), we see that

$$\begin{aligned} \frac{dM_t(\mathcal{U})}{M_t(\mathcal{U})} &= \frac{d\mathcal{U}(\mathbf{Y})}{\mathcal{U}(\mathbf{Y})} - \frac{d\mathcal{Z}(\mathbf{Y})}{\mathcal{Z}(\mathbf{Y})} + 4 \left(\frac{\partial_1 \mathcal{Z}(\mathbf{Y})}{\mathcal{Z}(\mathbf{Y})} \right)^2 - 4 \left(\frac{\partial_1 \mathcal{U}(\mathbf{Y})}{\mathcal{U}(\mathbf{Y})} \right) \left(\frac{\partial_1 \mathcal{Z}(\mathbf{Y})}{\mathcal{Z}(\mathbf{Y})} \right) \\ &= \left(\frac{2\partial_1 \mathcal{U}(\mathbf{Y})}{\mathcal{U}(\mathbf{Y})} - \frac{2\partial_1 \mathcal{Z}(\mathbf{Y})}{\mathcal{Z}(\mathbf{Y})} \right) dB_t + \frac{\mathcal{D}^{(1)} \mathcal{U}(\mathbf{Y})}{\mathcal{U}(\mathbf{Y})} dt. \end{aligned}$$

This implies that $M_t(\mathcal{U})$ is a local martingale if and only if $\mathcal{D}^{(1)} \mathcal{U} = 0$. \square

We now give the formula for the connection probabilities for the level lines of the GFF. To emphasize the main idea, we postpone some technical results (Proposition 5.10 and Corollary 6.10) to later sections.

Lemma 5.6. *We have*

$$P_\alpha = \frac{\mathcal{Z}_\alpha(x_1, \dots, x_{2N})}{\mathcal{Z}_{\text{GFF}}^{(N)}(x_1, \dots, x_{2N})} > 0 \quad \text{for all } \alpha \in \text{LP}_N, \quad \text{where } \mathcal{Z}_{\text{GFF}}^{(N)} = \sum_{\alpha \in \text{LP}_N} \mathcal{Z}_\alpha, \quad (5.6)$$

and $\{\mathcal{Z}_\alpha : \alpha \in \text{LP}\}$ is the collection of functions of Theorem 1.1 with $\kappa = 4$.

Proof. By Theorem 1.1, we have $\mathcal{Z}_\alpha > 0$ for all $\alpha \in \text{LP}_N$, so

$$0 < \frac{\mathcal{Z}_\alpha(x_1, \dots, x_{2N})}{\mathcal{Z}_{\text{GFF}}^{(N)}(x_1, \dots, x_{2N})} \leq 1, \quad \text{for all } (x_1, \dots, x_{2N}) \in \mathfrak{X}_{2N}. \quad (5.7)$$

We prove the assertion by induction on N . The base case $N = 0$ is a tautology: $\mathcal{Z}_\emptyset = 1 = \mathcal{Z}_{\text{GFF}}^{(0)}$. Assume that formula (5.6) is true for all $\hat{\alpha} \in \text{LP}_{N-1}$. Let $\alpha \in \text{LP}_N$. There exists $j \in \{1, 2, \dots, 2N\}$ such that $\{j, j+1\} \in \alpha$. Without loss of generality, we may assume that $\{1, 2\} \in \alpha$. Let η be the level line of the GFF h starting from x_1 , let T be its continuation threshold, and denote by W_t its driving function and $(g_t, t \geq 0)$ the corresponding family of conformal maps. Define, for $t < T$,

$$M_t(\mathcal{Z}_\alpha) = \frac{\mathcal{Z}_\alpha(W_t, g_t(x_2), \dots, g_t(x_{2N}))}{\mathcal{Z}_{\text{GFF}}^{(N)}(W_t, g_t(x_2), \dots, g_t(x_{2N}))}.$$

By Lemma 5.5, $M_t(\mathcal{Z}_\alpha)$ is a local martingale.

As $t \rightarrow T$, we know that $\eta(t) \rightarrow x_{2k}$ for some $k \in \{1, \dots, N\}$. First, we consider the case when $\eta(t) \rightarrow x_2$. By Corollary 6.10, we have almost surely on the event $\{\eta(T) = x_2\}$, as $t \rightarrow T$,

$$M_t(\mathcal{Z}_\alpha) = \frac{(g_t(x_2) - W_t)^{1/2} \mathcal{Z}_\alpha(W_t, g_t(x_2), \dots, g_t(x_{2N}))}{(g_t(x_2) - W_t)^{1/2} \mathcal{Z}_{\text{GFF}}^{(N)}(W_t, g_t(x_2), \dots, g_t(x_{2N}))} \rightarrow \frac{\mathcal{Z}_{\hat{\alpha}}(g_T(x_3), \dots, g_T(x_{2N}))}{\mathcal{Z}_{\text{GFF}}^{(N-1)}(g_T(x_3), \dots, g_T(x_{2N}))},$$

where $\hat{\alpha} = \alpha / \{1, 2\}$. Next, by Proposition 5.10, we have almost surely on the event $\{\eta(T) = x_{2k}\}$,

$$\lim_{t \rightarrow T} \frac{\mathcal{Z}_\alpha(W_t, g_t(x_2), \dots, g_t(x_{2N}))}{\mathcal{Z}_{\text{GFF}}^{(N)}(W_t, g_t(x_2), \dots, g_t(x_{2k}))} = 0.$$

In summary, we have almost surely,

$$M_T(\mathcal{Z}_\alpha) := \lim_{t \rightarrow T} M_t(\mathcal{Z}_\alpha) = \mathbb{1}_{\{\eta(T) = x_2\}} \frac{\mathcal{Z}_{\hat{\alpha}}(g_T(x_3), \dots, g_T(x_{2N}))}{\mathcal{Z}_{\text{GFF}}^{(N-1)}(g_T(x_3), \dots, g_T(x_{2N}))}.$$

By (5.7), $M_t(\mathcal{Z}_\alpha)$ is bounded, so Optional Stopping Theorem gives

$$\frac{\mathcal{Z}_\alpha}{\mathcal{Z}_{\text{GFF}}^{(N)}} = M_0(\mathcal{Z}_\alpha) = \mathbb{E}[M_T(\mathcal{Z}_\alpha)].$$

Combining with the induction hypothesis $P_{\hat{\alpha}} = \mathcal{Z}_{\hat{\alpha}} / \mathcal{Z}_{\text{GFF}}^{(N-1)}$, we obtain

$$\frac{\mathcal{Z}_\alpha}{\mathcal{Z}_{\text{GFF}}^{(N)}} = \mathbb{E}[\mathbb{1}_{\{\eta(T) = x_2\}} P_{\hat{\alpha}}(g_T(x_3), \dots, g_T(x_{2N}))]. \quad (5.8)$$

On the other hand, consider the level lines $(\eta_1, \eta_2, \dots, \eta_N)$ of the GFF h , where η_j is the level line starting from x_{2j-1} . Given $\eta := \eta_1$, on the event $\{\eta(T) = x_2\}$, the conditional law of (η_2, \dots, η_N) is that of the level lines of the GFF \hat{h} with alternating boundary conditions, where \hat{h} is h restricted to the unbounded component of $\mathbb{H} \setminus \eta$. Thus,

$$P_\alpha = \mathbb{E}[\mathbb{1}_{\{\eta(T) = x_2\}} P_{\hat{\alpha}}(g_T(x_3), \dots, g_T(x_{2N}))]. \quad (5.9)$$

Combining (5.8) and (5.9), we obtain $P_\alpha = \mathcal{Z}_\alpha / \mathcal{Z}_{\text{GFF}}^{(N)}$, which is what we sought to prove. \square

5.5 Marginal Probabilities and Proof of Theorem 1.4

Next we calculate the probability for one level line of the GFF to terminate at a given point. Again, we postpone some technical results to the next section.

Proposition 5.7. *For $l, k \in \{1, 2, \dots, 2N\}$ such that l is odd and k is even, the probability $P^{(l,k)}$ that the level line of the GFF starting from x_l terminates at x_k is given by*

$$P^{(l,k)} = \prod_{\substack{1 \leq j \leq 2N, \\ j \neq l, k}} \left(\frac{|x_j - x_l|}{|x_j - x_k|} \right)^{\delta_j}, \quad \text{where } \delta_j = (-1)^j.$$

Before proving the proposition, we observe that a special case follows by easy martingale arguments.

Lemma 5.8. *The conclusion in Proposition 5.7 holds for $k = l + 1$.*

Proof. To simplify notation, we assume $l = 1$; the other cases are similar. The level line $\eta := \eta_1$ started from x_1 is an $\text{SLE}_4(-2, +2, \dots, -2)$ process with force points (x_2, \dots, x_{2N}) . Let T be the continuation threshold of η . Define, for $t < T$,

$$M_t = \prod_{j=3}^{2N} \left(\frac{g_t(x_j) - W_t}{g_t(x_j) - g_t(x_2)} \right)^{\delta_j}.$$

By Lemma 5.4 with $S = \{1, 2\}$, M_t is a local martingale. Remark 5.1 gives for $j \in \{3, 4, \dots, 2N\}$ that

$$\left(\frac{g_t(x_{j+1}) - W_t}{g_t(x_{j+1}) - g_t(x_2)} \right) \left(\frac{g_t(x_j) - g_t(x_2)}{g_t(x_j) - W_t} \right) \leq 1,$$

so M_t is bounded: we have $0 \leq M_t \leq 1$ for $t < T$. Finally, as $t \rightarrow T$, we have almost surely $M_t \rightarrow 1$ when $\eta(t) \rightarrow x_2$, and Lemma 5.11 shows that $M_t \rightarrow 0$ when $\eta(t) \rightarrow x_{2k}$ for $k \in \{2, 3, \dots, N\}$. Therefore, Optional Stopping Theorem implies $P^{(1,2)} = \mathbb{P}[\eta(T) = x_2] = \mathbb{E}[M_T] = M_0$, as desired. \square

To prove the general case in Proposition 5.7, we use the following lemma.

Lemma 5.9. *For any $N \geq 2$ and $l, k \in \{1, 2, \dots, 2N\}$ with odd l and even k , the function $F_N^{(l,k)} : \mathfrak{X}_{2N} \rightarrow \mathbb{C}$,*

$$F_N^{(l,k)}(x_1, \dots, x_{2N}) := \mathcal{Z}_{\text{GFF}}^{(N)}(x_1, \dots, x_{2N}) \prod_{\substack{1 \leq j \leq 2N, \\ j \neq l, k}} \left(\frac{|x_j - x_l|}{|x_j - x_k|} \right)^{\delta_j}, \quad \text{where } \delta_j = (-1)^j, \quad (5.10)$$

belongs to the solution space \mathcal{S}_N defined in (2.8).

Proof. The function $F_N^{(l,k)}$ clearly satisfies the bound (2.7). Also, because $\mathcal{Z}_{\text{GFF}}^{(N)}$ satisfies (COV) (1.2) and the product $\prod_j \left(\frac{|x_j - x_l|}{|x_j - x_k|} \right)^{\delta_j}$ is conformally invariant, $F_N^{(l,k)}$ also satisfies (COV) (1.2). It remains to show (PDE) (1.1). Without loss of generality, we may assume $l = 1$. Combining Lemma 5.5 and Lemma 5.4 (with $S = \{1, k\}$), we see that $F_N^{(1,k)}$ satisfies (PDE) (1.1) as well. Thus, we have $F_N^{(1,k)} \in \mathcal{S}_N$. \square

Proof of Proposition 5.7. On the one hand, because by Lemma 5.9 the function $F_N^{(l,k)}$ defined in (5.10) belongs to the space \mathcal{S}_N , Proposition 4.6 allows us to write it in the form

$$F_N^{(l,k)} = \sum_{\alpha \in \text{LP}_N} c_\alpha \mathcal{Z}_\alpha, \quad \text{where } c_\alpha = \mathcal{L}_\alpha(F_N^{(l,k)}).$$

On the other hand, by Equation (1.6) in Theorem 1.4, we have

$$P^{(l,k)} = \sum_{\alpha \in \text{LP}_N : \{l,k\} \in \alpha} P_\alpha = \sum_{\alpha \in \text{LP}_N : \{l,k\} \in \alpha} \frac{\mathcal{Z}_\alpha}{\mathcal{Z}_{\text{GFF}}^{(N)}}.$$

Thus, it suffices to show that

$$\mathcal{L}_\alpha(F_N^{(l,k)}) = \mathbb{1}\{\{l,k\} \in \alpha\}. \quad (5.11)$$

Without loss of generality, we may assume that $l = 1$. We prove (5.11) by induction on N . For $N = 1$ the claim is clear. Assume that we have $\mathcal{L}_\beta(F_{N-1}^{(1,k)}) = \mathbb{1}\{\{1,k\} \in \beta\}$ for all $\beta \in \text{LP}_{N-1}$ and $k \in \{2, 4, \dots, 2N-2\}$. Let $\alpha \in \text{LP}_N$. Choose i such that $\{i, i+1\} \in \alpha$. We consider two cases.

Case 1: $i, i+1 \notin \{1, k\}$. By the property (4.6) of the function $\mathcal{Z}_{\text{GFF}}^{(N)}$, we have, for any $\xi \in (x_{i-1}, x_{i+2})$,

$$\begin{aligned} & \lim_{x_i, x_{i+1} \rightarrow \xi} (x_{i+1} - x_i)^{1/2} F_N^{(1,k)}(x_1, \dots, x_{2N}) \\ &= \lim_{x_i, x_{i+1} \rightarrow \xi} (x_{i+1} - x_i)^{1/2} \mathcal{Z}_{\text{GFF}}^{(N)}(x_1, \dots, x_{2N}) \prod_{\substack{1 \leq j \leq 2N, \\ j \neq 1, k}} \left(\frac{|x_j - x_1|}{|x_j - x_k|} \right)^{\delta_j} \\ &= \mathcal{Z}_{\text{GFF}}^{(N-1)}(x_1, \dots, x_{i-1}, x_{i+2}, \dots, x_{2N}) \prod_{\substack{1 \leq j \leq 2N, \\ j \neq 1, k, i, i+1}} \left(\frac{|x_j - x_1|}{|x_j - x_k|} \right)^{\delta_j} \\ &= F_{N-1}^{(1,k')}(x_1, \dots, x_{i-1}, x_{i+2}, \dots, x_{2N}), \end{aligned}$$

where $k' = k$ if $i > k$ and $k' = k - 2$ if $i < k$. Thus, choosing an allowable ordering for the links in α in such a way that $\{a_1, b_1\} = \{i, i+1\}$, the induction hypothesis shows that

$$\mathcal{L}_\alpha(F_N^{(1,k)}) = \mathcal{L}_{\alpha/\{i, i+1\}}(F_{N-1}^{(1,k')}) = \mathbb{1}\{\{1, k'\} \in \alpha/\{i, i+1\}\} = \mathbb{1}\{\{1, k\} \in \alpha\}.$$

Case 2: $i \in \{1, k\}$ or $i+1 \in \{1, k\}$. Then we necessarily have $\{1, k\} \notin \alpha$. By symmetry, it suffices to treat the case $i = 1$. Then we have, for any $\xi \in (x_1, x_2)$,

$$\begin{aligned} & \lim_{x_1, x_2 \rightarrow \xi} (x_2 - x_1)^{1/2} F_N^{(1,k)}(x_1, \dots, x_{2N}) \\ &= \lim_{x_1, x_2 \rightarrow \xi} (x_2 - x_1)^{1/2} \mathcal{Z}_{\text{GFF}}^{(N)}(x_1, \dots, x_{2N}) \prod_{\substack{1 \leq j \leq 2N, \\ j \neq 1, 2, k}} \left(\frac{|x_j - x_1|}{|x_j - x_k|} \right)^{\delta_j} \times \frac{|x_2 - x_1|}{|x_2 - x_k|} = 0. \end{aligned}$$

This finishes the proof. \square

Collecting the results from this section and Section 5.4, we now prove Theorem 1.4.

Theorem 1.4. *Consider multiple level lines of the GFF on \mathbb{H} with alternating boundary conditions. For any $\alpha \in \text{LP}_N$, the probability $P_\alpha := \mathbb{P}[\mathcal{A} = \alpha]$ is strictly positive. Conditioned on the event $\{\mathcal{A} = \alpha\}$, the collection $(\eta_1, \dots, \eta_N) \in X_0^\alpha(\mathbb{H}; x_1, \dots, x_{2N})$ is the global N -SLE₄ associated to α constructed in Theorem 1.3. The connection probabilities are explicitly given by*

$$P_\alpha = \frac{\mathcal{Z}_\alpha(x_1, \dots, x_{2N})}{\mathcal{Z}_{\text{GFF}}^{(N)}(x_1, \dots, x_{2N})}, \quad \text{for all } \alpha \in \text{LP}_N, \quad \text{where } \mathcal{Z}_{\text{GFF}}^{(N)} := \sum_{\alpha \in \text{LP}_N} \mathcal{Z}_\alpha \quad (1.6)$$

and \mathcal{Z}_α are the functions of Theorem 1.1 with $\kappa = 4$. For $l, k \in \{1, 2, \dots, 2N\}$, where l is odd and k is even, the probability that the level line of the GFF starting from x_l terminates at x_k is given by

$$P^{(l,k)} = \prod_{\substack{1 \leq j \leq 2N, \\ j \neq l, k}} \left(\frac{|x_j - x_l|}{|x_j - x_k|} \right)^{\delta_j}, \quad \text{where } \delta_j = (-1)^j. \quad (1.7)$$

Proof. By Lemma 5.6, the connection probabilities $P_\alpha := \mathbb{P}[\mathcal{A} = \alpha]$ are given by (1.6) and they are strictly positive. Conditioned on the event $\{\mathcal{A} = \alpha\}$, we have $(\eta_1, \dots, \eta_N) \in X_0^\alpha(\mathbb{H}; x_1, \dots, x_{2N})$ and its law is a global N -SLE₄ associated to α : for each $j \in \{1, \dots, N\}$, the conditional law of η_j given $\{\eta_1, \dots, \eta_{j-1}, \eta_{j+1}, \dots, \eta_N\}$ is the same as the law of the level line of the GFF in $\hat{\Omega}_j$ with Dobrushin boundary conditions, which is that of the chordal SLE₄. This global N -SLE₄ is the global N -SLE₄ constructed in Theorem 1.3, since, by arguments presented in [MS16b, Theorem 4.1] and [Wu17, Remark 4.4], the global N -SLE₄ associated to α is unique. Finally, Proposition 5.7 proves (1.7). \square

5.6 Technical Lemmas

The purpose of this section is to prove the technicalities that were omitted before. The main result of this section is the following.

Proposition 5.10. *Let $\alpha = \{\{a_1, b_1\}, \dots, \{a_N, b_N\}\} \in \text{LP}_N$ and suppose that $\{1, 2\} \in \alpha$. Fix an index $k \in \{2, 3, \dots, N\}$ and real points $x_1 < x_2 < \dots < x_{2N}$. Suppose η is a continuous simple curve in \mathbb{H} starting from x_1 and terminating at x_{2k} at time T , which hits \mathbb{R} only at $\{x_1, x_{2k}\}$. Let $(W_t, 0 \leq t \leq T)$ be its Loewner driving function and $(g_t, 0 \leq t \leq T)$ the corresponding conformal maps. Then we have*

$$\lim_{t \rightarrow T} \frac{\mathcal{Z}_\alpha(W_t, g_t(x_2), \dots, g_t(x_{2N}))}{\mathcal{Z}_{\text{GFF}}^{(N)}(W_t, g_t(x_2), \dots, g_t(x_{2N}))} = 0. \quad (5.12)$$

We prove Proposition 5.10 in the end of this section, using the following lemmas.

Lemma 5.11. *Fix an index $k \in \{2, 3, \dots, N\}$ and real points $x_1 < x_2 < \dots < x_{2k}$. Suppose η is a continuous simple curve in \mathbb{H} starting from x_1 and terminating at x_{2k} at time T , which hits \mathbb{R} only at $\{x_1, x_{2k}\}$. Let $(W_t, 0 \leq t \leq T)$ be its Loewner driving function and $(g_t, 0 \leq t \leq T)$ the corresponding conformal maps. Then we have*

$$\lim_{t \rightarrow T} \prod_{j=3}^{2k} \left(\frac{g_t(x_j) - W_t}{g_t(x_j) - g_t(x_2)} \right)^{(-1)^j} = 0.$$

Proof. For all odd $j \in \{3, 5, \dots, 2k-3\}$, Remark 5.1 shows that

$$0 \leq \frac{(g_t(x_j) - g_t(x_2))(g_t(x_{j+1}) - W_t)}{(g_t(x_j) - W_t)(g_t(x_{j+1}) - g_t(x_2))} \leq 1.$$

Combining this with Lemma 5.2, we see that, when $t \rightarrow T$,

$$0 \leq \prod_{j=3}^{2N} \left(\frac{g_t(x_j) - W_t}{g_t(x_j) - g_t(x_2)} \right)^{(-1)^j} \leq \frac{(g_t(x_{2k-1}) - g_t(x_2))(g_t(x_{2k}) - W_t)}{(g_t(x_{2k-1}) - W_t)(g_t(x_{2k}) - g_t(x_2))} \rightarrow 0,$$

as claimed. \square

For any $\alpha = \{\{a_1, b_1\}, \dots, \{a_N, b_N\}\} \in \text{LP}_N$ and $k \in \{1, \dots, N\}$, define the function

$$F_\alpha^{(k)}(x_1, \dots, x_{2N}) := \left(\prod_{j=1}^N |x_{b_j} - x_{a_j}|^{1/2} \times \mathcal{Z}_{\text{GFF}}^{(k)}(x_1, \dots, x_{2k}) \right)^{-1}. \quad (5.13)$$

Lemma 5.12. *Let $\alpha \in \text{LP}_N$ and suppose that $\{1, 2\} \in \alpha$. Then for all $k \in \{1, \dots, N\}$, we have*

$$F_\alpha^{(k)}(x_1, x_2, x_3, \dots, x_{2N}) = \prod_{j=3}^{2k} \left(\frac{x_j - x_1}{x_j - x_2} \right)^{\frac{1}{2}(-1)^j} \times F_{\hat{\alpha}}^{(k-1)}(x_3, x_4, \dots, x_{2N}),$$

where $\hat{\alpha} = \alpha / \{1, 2\}$.

Proof. This follows immediately from the definition (5.13) of $F_\alpha^{(k)}$. \square

Lemma 5.13. *For any $\alpha \in \text{LP}_N$, $k \in \{1, \dots, N\}$, and $\xi < x_{2k+1} < \dots < x_{2N}$, we have*

$$\lim_{\substack{\tilde{x}_1, \dots, \tilde{x}_{2k} \rightarrow \xi, \\ \tilde{x}_i \rightarrow x_i \text{ for } 2k < i \leq 2N}} F_\alpha^{(k)}(\tilde{x}_1, \dots, \tilde{x}_{2N}) < \infty. \quad (5.14)$$

Proof. We prove the claim by induction on N . It is clear for $N = 1$, as $F_\emptyset^{(0)} = 1$. Assume the property

$$\lim_{\substack{\tilde{x}_1, \dots, \tilde{x}_{2\ell} \rightarrow y, \\ \tilde{x}_i \rightarrow x_i \text{ for } 2\ell < i \leq 2N-2}} F_\beta^{(\ell)}(\tilde{x}_1, \dots, \tilde{x}_{2N-2}) < \infty$$

holds for all $\beta \in \text{LP}_{N-1}$, $\ell \in \{1, \dots, N-1\}$ and $y < x_{2k+1} < \dots < x_{2N-2}$. Fix $\alpha \in \text{LP}_N$, $k \in \{1, \dots, N\}$, and $\xi < x_{2k+1} < \dots < x_{2N}$. Choose j such that $\{j, j+1\} \in \alpha$. We consider three cases.

Case 1: $j+1 \leq 2k$. In this case, by Lemma 5.12, we have

$$F_\alpha^{(k)}(\tilde{x}_1, \dots, \tilde{x}_{2N}) = \prod_{\substack{1 \leq i \leq 2k, \\ i \neq j, j+1}} \left| \frac{\tilde{x}_i - \tilde{x}_j}{\tilde{x}_i - \tilde{x}_{j+1}} \right|^{\frac{1}{2}(-1)^{i+j+1}} F_{\alpha/\{j, j+1\}}^{(k-1)}(\tilde{x}_1, \dots, \tilde{x}_{j-1}, \tilde{x}_{j+2}, \dots, \tilde{x}_{2N}).$$

Using Remark 5.1, we see that if j is odd, then we have

$$\prod_{\substack{1 \leq i \leq 2k, \\ i \neq j, j+1}} \left| \frac{\tilde{x}_i - \tilde{x}_j}{\tilde{x}_i - \tilde{x}_{j+1}} \right|^{(-1)^{i+j+1}} = \prod_{\substack{1 \leq i \leq k, \\ i \neq (j+1)/2}} \left| \frac{(\tilde{x}_{2i-1} - \tilde{x}_{j+1})(\tilde{x}_{2i} - \tilde{x}_j)}{(\tilde{x}_{2i-1} - \tilde{x}_j)(\tilde{x}_{2i} - \tilde{x}_{j+1})} \right| \leq 1,$$

and if j is even, then we have

$$\prod_{\substack{1 \leq i \leq 2k, \\ i \neq j, j+1}} \left| \frac{\tilde{x}_i - \tilde{x}_j}{\tilde{x}_i - \tilde{x}_{j+1}} \right|^{(-1)^{i+j+1}} = \left| \frac{(\tilde{x}_1 - \tilde{x}_j)(\tilde{x}_{2k} - \tilde{x}_{j+1})}{(\tilde{x}_1 - \tilde{x}_{j+1})(\tilde{x}_{2k} - \tilde{x}_j)} \right| \prod_{\substack{1 \leq i < k, \\ i \neq j/2}} \left| \frac{(\tilde{x}_{2i} - \tilde{x}_{j+1})(\tilde{x}_{2i+1} - \tilde{x}_j)}{(\tilde{x}_{2i} - \tilde{x}_j)(\tilde{x}_{2i+1} - \tilde{x}_{j+1})} \right| \leq 1.$$

Thus, we have

$$F_\alpha^{(k)}(\tilde{x}_1, \dots, \tilde{x}_{2N}) \leq F_{\alpha/\{j, j+1\}}^{(k-1)}(\tilde{x}_1, \dots, \tilde{x}_{j-1}, \tilde{x}_{j+2}, \dots, \tilde{x}_{2N}),$$

so by the induction hypothesis, $F_\alpha^{(k)}$ remains finite in the limit (5.14).

Case 2: $j > 2k$. In this case, we have

$$F_\alpha^{(k)}(\tilde{x}_1, \dots, \tilde{x}_{2N}) = (\tilde{x}_{j+1} - \tilde{x}_j)^{-1/2} F_{\alpha/\{j, j+1\}}^{(k)}(\tilde{x}_1, \dots, \tilde{x}_{j-1}, \tilde{x}_{j+2}, \dots, \tilde{x}_{2N}),$$

which by the induction hypothesis remains finite in the limit (5.14).

Case 3: $j = 2k$. In this case, we have

$$F_\alpha^{(k)}(\tilde{x}_1, \dots, \tilde{x}_{2N}) = \left(\frac{\tilde{x}_{2k} - \tilde{x}_{2k-1}}{\tilde{x}_{2k+1} - \tilde{x}_{2k}} \right)^{1/2} \times \prod_{i=1}^{2k-2} \left(\frac{\tilde{x}_{2k-1} - \tilde{x}_i}{\tilde{x}_{2k} - \tilde{x}_i} \right)^{\frac{1}{2}(-1)^i} \times F_{\alpha/\{j, j+1\}}^{(k-1)}(\tilde{x}_1, \dots, \tilde{x}_{j-1}, \tilde{x}_{j+2}, \dots, \tilde{x}_{2N}).$$

By Remark 5.1, we have

$$\prod_{i=1}^{2k-2} \left(\frac{\tilde{x}_{2k-1} - \tilde{x}_i}{\tilde{x}_{2k} - \tilde{x}_i} \right)^{(-1)^i} = \prod_{i=1}^{k-1} \frac{(\tilde{x}_{2k} - \tilde{x}_{2i-1})(\tilde{x}_{2k-1} - \tilde{x}_{2i})}{(\tilde{x}_{2k-1} - \tilde{x}_{2i-1})(\tilde{x}_{2k} - \tilde{x}_{2i})} \leq 1.$$

By the induction hypothesis, the limit (5.14) of $F_{\alpha/\{j, j+1\}}^{(k-1)}$ is finite, so we see that $F_\alpha^{(k)}$ also remains finite in the limit (5.14) (in fact, the limit of $F_\alpha^{(k)}$ is zero in this case). This completes the proof. \square

Proof of Proposition 5.10. Write $\mathcal{Z}_\alpha/\mathcal{Z}_{\text{GFF}}^{(N)} = \left(\mathcal{Z}_{\text{GFF}}^{(k)}/\mathcal{Z}_{\text{GFF}}^{(N)}\right) \left(\mathcal{Z}_\alpha/\mathcal{Z}_{\text{GFF}}^{(k)}\right)$. Corollary 4.12 shows that in the limit (5.12), we have $\mathcal{Z}_{\text{GFF}}^{(N)}/\mathcal{Z}_{\text{GFF}}^{(k)} \rightarrow \mathcal{Z}_{\text{GFF}}^{(N-k)} > 0$. Thus, it suffices to show that $\mathcal{Z}_\alpha/\mathcal{Z}_{\text{GFF}}^{(k)} \rightarrow 0$ in this limit. Using the bound (1.4), we see that $\mathcal{Z}_\alpha/\mathcal{Z}_{\text{GFF}}^{(k)} \leq F_\alpha^{(k)}$, where $F_\alpha^{(k)}$ is the function defined in (5.13). Now, combining Lemmas 5.11 – 5.13, we obtain, as $t \rightarrow T$,

$$\begin{aligned} \frac{\mathcal{Z}_\alpha(W_t, g_t(x_2), \dots, g_t(x_{2N}))}{\mathcal{Z}_{\text{GFF}}^{(k)}(W_t, g_t(x_2), \dots, g_t(x_{2N}))} &\leq F_\alpha^{(k)}(W_t, g_t(x_2), \dots, g_t(x_{2N})) \\ &= \prod_{j=3}^{2k} \left(\frac{g_t(x_j) - W_t}{g_t(x_j) - g_t(x_2)} \right)^{\frac{1}{2}(-1)^j} \times F_{\hat{\alpha}}^{(k-1)}(g_t(x_3), \dots, g_t(x_{2N})) \rightarrow 0, \end{aligned}$$

where $\hat{\alpha} = \alpha/\{1, 2\}$. This concludes the proof. \square

6 Pure Partition Functions for Multiple SLE₄

In the previous section, we solved the connection probabilities for the level lines of the GFF in terms of the multiple SLE₄ pure partition functions. On the other hand, we constructed the multiple SLE _{κ} pure partition functions for all $\kappa \in (0, 4]$ in Section 3, see Equation (3.7). The purpose of this section is to give another, algebraic formula for them in the case of $\kappa = 4$. Our goal is to prove Theorem 1.5:

Theorem 1.5. *Let $\kappa = 4$. Then, the functions $\{\mathcal{Z}_\alpha : \alpha \in \text{LP}\}$ of Theorem 1.1 can be written as*

$$\mathcal{Z}_\alpha(x_1, \dots, x_{2N}) = \sum_{\beta \in \text{LP}_N} \mathcal{M}_{\alpha, \beta}^{-1} \mathcal{U}_\beta(x_1, \dots, x_{2N}), \quad (1.8)$$

where \mathcal{U}_β are explicit functions defined in Equation (6.2) and the coefficients $\mathcal{M}_{\alpha, \beta}^{-1} \in \mathbb{Z}$ are given in Proposition 6.6 in Section 6.

The main virtue of the formula (1.8) in Theorem 1.5 is that for each $\alpha \in \text{LP}_N$, it expresses the pure partition function \mathcal{Z}_α as a finite sum of well-behaved functions \mathcal{U}_β , for $\beta \in \text{LP}_N$, with explicit integer coefficients that enumerate certain combinatorial objects only depending on α and β (see Proposition 6.6 for details). Such combinatorial enumerations have been studied e.g. in [KW11a, KW11b, KKP17] and they have many desirable properties which can be used in analyzing the pure partition functions. As an example of this, we verify in Section 6.1 that the decay of the rainbow connection probability agrees with the boundary arm exponents (or (half-)watermelon exponents) appearing in the physics literature.

This kind of algebraic formulas for connection probabilities were first derived by R. Kenyon and D. Wilson [KW11a, KW11b] in the context of crossing probabilities in discrete models (in a very general setup, which includes the Loop-Erased Random Walk, $\kappa = 2$; and the double-dimer model, $\kappa = 4$). In [KKP17], A. Karrila, K. Kytölä, and E. Peltola studied the scaling limits of connection probabilities of Loop-Erased Random Walks and identified them with the multiple SLE₂ pure partition functions.

We give here an outline of the proof of Theorem 1.5. It relies on straightforward calculations and combinatorics presented in Sections 6.3 and 6.4, and the uniqueness result of Lemma 4.5 from Section 3.

Proof of Theorem 1.5. By definition, the functions appearing on the right hand side of (1.8) satisfy the normalization $\mathcal{Z}_\emptyset = 1$ and the bound (2.7) — this follows immediately from the definition (6.2) of \mathcal{U}_α , since the coefficients $\mathcal{M}_{\alpha, \beta}^{-1}$ do not depend on (x_1, \dots, x_{2N}) . These functions also satisfy properties (PDE) (1.1) and (COV) (1.2) with $\kappa = 4$, by linearity and the corresponding properties of \mathcal{U}_α from Lemmas 6.3 and 6.4 below. Finally, they also have the asymptotics property (ASY) (1.3) with $\kappa = 4$, by Lemma 6.9 below. Thus, Lemma 4.5 shows that they must be equal to \mathcal{Z}_α . \square

We point out that in the proof of Theorem 1.5, it is not difficult to check properties (PDE) (1.1) and (COV) (1.2) with $\kappa = 4$ — this is a direct calculation done in Section 6.2. The main step of the proof is establishing the asymptotics of (1.8), which we do by combinatorial calculations in Section 6.4.

In the formula for the pure partition functions \mathcal{Z}_α , we use certain auxiliary functions \mathcal{U}_α , which in the conformal field theory (CFT) literature are known as conformal blocks [BPZ84a, FFK89, DFMS97, Rib14, FP17]¹. In Section 6.5, we give a relation between these conformal blocks and level lines of the GFF. For each link pattern $\alpha \in \text{LP}_N$, we define the conformal block function $\mathcal{U}_\alpha: \mathfrak{X}_{2N} \rightarrow \mathbb{R}_{>0}$ as follows. We write α as an ordered collection

$$\alpha = \{\{a_1, b_1\}, \dots, \{a_N, b_N\}\}, \quad \text{where } a_1 < a_2 < \dots < a_N \text{ and } a_j < b_j \text{ for all } j \in \{1, \dots, N\}. \quad (6.1)$$

Then, we set $\vartheta_\alpha(i, i) := 0$ and

$$\mathcal{U}_\alpha(x_1, \dots, x_{2N}) := \prod_{1 \leq i < j \leq 2N} (x_j - x_i)^{\frac{1}{2}\vartheta_\alpha(i, j)}, \quad (6.2)$$

$$\text{where } \vartheta_\alpha(i, j) := \begin{cases} +1 & \text{if } i, j \in \{a_1, a_2, \dots, a_N\} \text{ or } i, j \in \{b_1, b_2, \dots, b_N\} \\ -1 & \text{otherwise.} \end{cases}$$

Notice that when $\alpha = \sqcap \sqcap_N := \{\{1, 2\}, \{3, 4\}, \dots, \{2N-1, 2N\}\}$ is the completely unnested link pattern, then formula (6.2) equals that of the total partition function from Lemma 4.11, so $\mathcal{U}_{\sqcap \sqcap_N} = \mathcal{Z}_{\text{GFF}}^{(N)}$. Also, it follows from Proposition 6.6 below that when $\alpha = \sqcup \sqcup_N := \{\{1, 2N\}, \{2, 2N-1\}, \dots, \{N-1, N\}\}$ is the rainbow link pattern, then we have $\mathcal{U}_{\sqcup \sqcup_N} = \mathcal{Z}_{\sqcup \sqcup_N}$. In general, the conformal block functions \mathcal{U}_α and the pure partition functions \mathcal{Z}_α , for $\alpha \in \text{LP}_N$, form two linearly independent sets that are related by a non-trivial change of basis. The coefficients $\mathcal{M}_{\alpha, \beta}^{-1}$ in Theorem 1.5 are matrix elements of this invertible change of basis matrix, see Proposition 6.6.

Next, we list some examples of the explicit formulas for \mathcal{Z}_α given by (1.8) in Theorem 1.5.

- $N = 1$ is trivial: $\mathcal{Z}_{\sqcup \sqcup}(x_1, x_2) = (x_2 - x_1)^{-1/2} = \mathcal{U}_{\sqcup \sqcup}(x_1, x_2)$.
- $N = 2$: there are two link patterns, and denoting $x_{ji} := x_j - x_i$, we have (see also Table 1)

$$\mathcal{Z}_{\overbrace{\sqcup \sqcup}^{\text{nested}}}(x_1, x_2, x_3, x_4) = \mathcal{U}_{\overbrace{\sqcup \sqcup}^{\text{nested}}}(x_1, x_2, x_3, x_4) = \left(\frac{x_{43}x_{21}}{x_{41}x_{31}x_{42}x_{32}} \right)^{1/2},$$

$$\mathcal{Z}_{\underbrace{\sqcup \sqcup}_{\text{rainbow}}}(x_1, x_2, x_3, x_4) = \mathcal{U}_{\underbrace{\sqcup \sqcup}_{\text{rainbow}}}(x_1, x_2, x_3, x_4) - \mathcal{U}_{\overbrace{\sqcup \sqcup}^{\text{nested}}}(x_1, x_2, x_3, x_4) = \left(\frac{x_{41}x_{32}}{x_{31}x_{21}x_{43}x_{42}} \right)^{1/2}.$$

Note that these formulas are consistent with Lemma 5.3.

- $N = 3$: there are five link patterns, and we have (see also Table 2)

$$\begin{aligned} \mathcal{Z}_{\overbrace{\sqcup \sqcup \sqcup}^{\text{nested}}} &= \mathcal{U}_{\overbrace{\sqcup \sqcup \sqcup}^{\text{nested}}}, \\ \mathcal{Z}_{\overbrace{\sqcup \sqcup \sqcup}^{\text{nested}} \text{ (middle)}} &= \mathcal{U}_{\overbrace{\sqcup \sqcup \sqcup}^{\text{nested}} \text{ (middle)}} - \mathcal{U}_{\overbrace{\sqcup \sqcup \sqcup}^{\text{nested}} \text{ (left)}}, \\ \mathcal{Z}_{\overbrace{\sqcup \sqcup \sqcup}^{\text{nested}} \text{ (right)}} &= \mathcal{U}_{\overbrace{\sqcup \sqcup \sqcup}^{\text{nested}} \text{ (right)}} - \mathcal{U}_{\overbrace{\sqcup \sqcup \sqcup}^{\text{nested}} \text{ (middle)}} + \mathcal{U}_{\overbrace{\sqcup \sqcup \sqcup}^{\text{nested}} \text{ (left)}}, \\ \mathcal{Z}_{\overbrace{\sqcup \sqcup \sqcup}^{\text{nested}} \text{ (left)}} &= \mathcal{U}_{\overbrace{\sqcup \sqcup \sqcup}^{\text{nested}} \text{ (left)}} - \mathcal{U}_{\overbrace{\sqcup \sqcup \sqcup}^{\text{nested}} \text{ (middle)}} + \mathcal{U}_{\overbrace{\sqcup \sqcup \sqcup}^{\text{nested}} \text{ (right)}}, \\ \mathcal{Z}_{\underbrace{\sqcup \sqcup \sqcup}_{\text{rainbow}}} &= \mathcal{U}_{\underbrace{\sqcup \sqcup \sqcup}_{\text{rainbow}}} - \mathcal{U}_{\overbrace{\sqcup \sqcup \sqcup}^{\text{nested}} \text{ (middle)}} - \mathcal{U}_{\overbrace{\sqcup \sqcup \sqcup}^{\text{nested}} \text{ (left)}} + \mathcal{U}_{\overbrace{\sqcup \sqcup \sqcup}^{\text{nested}} \text{ (right)}} - 2\mathcal{U}_{\overbrace{\sqcup \sqcup \sqcup}^{\text{nested}} \text{ (left)}}. \end{aligned}$$

¹ The PDE system (1.1) is related to certain quantities being martingales for SLE_4 type curves (see Lemma 5.5). These partial differential equations arise in conformal field theory as well, from degenerate representations of the Virasoro algebra, see e.g. [DFMS97, Rib14]. The connection of the SLE_κ with conformal field theory is now well-known [BB03, FW03, BB04, Fri04, FK04, BBK05, Kyt07]: martingales for SLE_κ curves correspond with correlations in a CFT of central charge $c = (3\kappa - 8)(6 - \kappa)/2\kappa$. In that sense, it is natural that the conformal block functions satisfy (PDE) (1.1) — they are (chiral) correlation functions of a CFT with central charge $c = 1$. Also the asymptotics property (ASY) (1.3) for \mathcal{Z}_α can be related to fusion in CFT [Car89, BBK05, Dub15b, KP16].

6.1 Decay Properties of Pure Partition Functions with $\kappa = 4$

Consider the rainbow link pattern $\underline{\mathbb{m}}_N$ (see Figure 3.1). We prove next that, when its first N variables (or both the first N and the last N variables) tend together, the decay of the pure partition function $\mathcal{Z}_{\underline{\mathbb{m}}_N}$ agrees with the predictions from the physics literature for certain surface critical exponents [Car84, DS87, Nie87, Wer04, Wu16], known as boundary arm exponents (or (half-)watermelon exponents).

Proposition 6.1. *The rainbow pure partition function has the following decay as its N first variables tend together:*

$$\mathcal{Z}_{\underline{\mathbb{m}}_N}(\epsilon, 2\epsilon, \dots, N\epsilon, 1, 2, \dots, N) \sim \epsilon^{N(N-1)/4} \quad \text{as } \epsilon \rightarrow 0.$$

The total partition function $\mathcal{Z}_{\text{GFF}}^{(N)}$ has the decay

$$\mathcal{Z}_{\text{GFF}}^{(N)}(\epsilon, 2\epsilon, \dots, N\epsilon, 1, 2, \dots, N) \sim \begin{cases} \epsilon^{-N/4} & \text{if } N \text{ is even} \\ \epsilon^{-(N-1)/4} & \text{if } N \text{ is odd} \end{cases} \quad \text{as } \epsilon \rightarrow 0.$$

Proof. Proposition 6.6 below shows that $\mathcal{Z}_{\underline{\mathbb{m}}_N} = \mathcal{U}_{\underline{\mathbb{m}}_N}$. Now, it is clear from the definition (6.2) of $\mathcal{U}_{\underline{\mathbb{m}}_N}$ as a product that the decay from the first N variables $x_j = j\epsilon$, for $j \in \{1, \dots, N\}$, is ϵ^p , where the power can be read off from Equation (6.2): $p = \frac{N(N-1)}{2} \times \frac{1}{2}(+1) = \frac{N(N-1)}{4}$. Similarly, by the formula (4.10) of Lemma 4.11, the decay of the total partition function $\mathcal{Z}_{\text{GFF}}^{(N)}$ is also of type $\epsilon^{p'}$. To find out the power, we collect the exponents from the differences of the variables $x_j = j\epsilon$, for $j \in \{1, \dots, N\}$, in (4.10):

$$p' = \sum_{1 \leq k < l \leq N} \frac{1}{2}(-1)^{l-k} = \frac{1}{2} \sum_{k=1}^{N-1} \sum_{m=1}^{N-k} (-1)^m = \begin{cases} -N/4 & \text{if } N \text{ is even} \\ -(N-1)/4 & \text{if } N \text{ is odd.} \end{cases}$$

□

We see from Proposition 6.1 that for the level lines of the GFF, the connection probability associated to the rainbow link pattern (given in Theorem 1.4) has the decay

$$P_{\underline{\mathbb{m}}_N} = \frac{\mathcal{Z}_{\underline{\mathbb{m}}_N}(\epsilon, 2\epsilon, \dots, N\epsilon, 1, 2, \dots, N)}{\mathcal{Z}_{\text{GFF}}^{(N)}(\epsilon, 2\epsilon, \dots, N\epsilon, 1, 2, \dots, N)} \sim \epsilon^{\alpha_N^+}, \quad \text{as } \epsilon \rightarrow 0, \quad \text{where } \alpha_N^+ = \begin{cases} N^2/4 & \text{if } N \text{ is even} \\ (N^2 - 1)/4 & \text{if } N \text{ is odd.} \end{cases}$$

The exponent α_N^+ agrees with the SLE_4 boundary arm exponents derived in [Wu16, Proposition 3.1].

Corollary 6.2. *The rainbow pure partition function has the following decay as both its N first variables and its N last variables tend together:*

$$\mathcal{Z}_{\underline{\mathbb{m}}_N}(\epsilon, 2\epsilon, \dots, N\epsilon, 1 + \epsilon, 1 + 2\epsilon, \dots, 1 + N\epsilon) \sim \epsilon^{N(N-1)/2} \quad \text{as } \epsilon \rightarrow 0.$$

The total partition function $\mathcal{Z}_{\text{GFF}}^{(N)}$ has the decay

$$\mathcal{Z}_{\text{GFF}}^{(N)}(\epsilon, 2\epsilon, \dots, N\epsilon, 1 + \epsilon, 1 + 2\epsilon, \dots, 1 + N\epsilon) \sim \begin{cases} \epsilon^{-N/2} & \text{if } N \text{ is even} \\ \epsilon^{-(N-1)/2} & \text{if } N \text{ is odd} \end{cases} \quad \text{as } \epsilon \rightarrow 0.$$

Proof. Because the two sets $\{\epsilon, 2\epsilon, \dots, N\epsilon\}$ and $\{1 + \epsilon, 1 + 2\epsilon, \dots, 1 + N\epsilon\}$ of variables tend to 0 and 1, respectively, we only have to add up the power-law decay of Proposition 6.1 for both. □

6.2 First Properties of the Conformal Blocks

Now we verify properties (PDE) (1.1) and (COV) (1.2) with $\kappa = 4$ for \mathcal{U}_α , used in the proof of Theorem 1.5.

Lemma 6.3. *The functions $\mathcal{U}_\alpha: \mathfrak{X}_{2N} \rightarrow \mathbb{R}_{>0}$ defined in equation (6.2) satisfy (PDE) (1.1) with $\kappa = 4$.*

Proof. For simplicity, we denote $x_{ij} = x_i - x_j$ and $\mathbf{x} = (x_1, \dots, x_{2N}) \in \mathfrak{X}_{2N}$. We need to show that for fixed $i \in \{1, \dots, 2N\}$, we have

$$2 \frac{\partial_i^2 \mathcal{U}_\alpha(\mathbf{x})}{\mathcal{U}_\alpha(\mathbf{x})} + \sum_{j \neq i} \left(\frac{2}{x_{ji}} \frac{\partial_j \mathcal{U}_\alpha(\mathbf{x})}{\mathcal{U}_\alpha(\mathbf{x})} - \frac{1}{2x_{ji}^2} \right) = 0. \quad (6.3)$$

The terms with derivatives are

$$2 \frac{\partial_i^2 \mathcal{U}_\alpha(\mathbf{x})}{\mathcal{U}_\alpha(\mathbf{x})} = \frac{1}{2} \sum_{j, k \neq i} \frac{\vartheta_\alpha(i, j) \vartheta_\alpha(i, k)}{x_{ij} x_{ik}} - \sum_{j \neq i} \frac{\vartheta_\alpha(i, j)}{x_{ij}^2} \quad \text{and} \quad \sum_{j \neq i} \frac{2}{x_{ji}} \frac{\partial_j \mathcal{U}_\alpha(\mathbf{x})}{\mathcal{U}_\alpha(\mathbf{x})} = \sum_{j \neq i} \frac{1}{x_{ji}} \sum_{k \neq j} \frac{\vartheta_\alpha(j, k)}{x_{jk}}.$$

Using this, the left hand side of the PDE (6.3) becomes

$$\frac{1}{2} \sum_{j, k \neq i} \frac{\vartheta_\alpha(i, j) \vartheta_\alpha(i, k)}{x_{ij} x_{ik}} - \sum_{j \neq i} \frac{\vartheta_\alpha(i, j)}{x_{ij}^2} + \sum_{j \neq i} \frac{1}{x_{ji}} \sum_{k \neq j} \frac{\vartheta_\alpha(j, k)}{x_{jk}} - \sum_{j \neq i} \frac{1}{2x_{ji}^2}. \quad (6.4)$$

The last term of (6.4) is canceled by the case $k = j$ in the first term, and the second term of (6.4) is canceled by the case $k = i$ in the third term. We are left with

$$\sum_{j \neq i} \frac{1}{x_{ji}} \sum_{k \neq i, j} \left(\frac{\vartheta_\alpha(j, k)}{x_{jk}} - \frac{\vartheta_\alpha(i, j) \vartheta_\alpha(i, k)}{2x_{ik}} \right). \quad (6.5)$$

For a pair (j, k) such that $j \neq i$ and $k \neq i, j$, combining the terms where j and k are interchanged, we get a term of the form

$$\left(\frac{\vartheta_\alpha(j, k)}{x_{ji} x_{jk}} + \frac{\vartheta_\alpha(k, j)}{x_{ki} x_{kj}} \right) + \frac{\vartheta_\alpha(i, j) \vartheta_\alpha(i, k)}{x_{ji} x_{ki}} = \frac{\vartheta_\alpha(j, k)}{x_{ji} x_{ik}} + \frac{\vartheta_\alpha(i, j) \vartheta_\alpha(i, k)}{x_{ji} x_{ki}} = \frac{\vartheta_\alpha(j, k) - \vartheta_\alpha(i, j) \vartheta_\alpha(i, k)}{x_{ji} x_{ik}}.$$

It remains to notice that the numbers ϑ_α defined in (6.2) satisfy the identity $\vartheta_\alpha(j, k) - \vartheta_\alpha(i, j) \vartheta_\alpha(i, k) = 0$ for all i, j and k . This shows that (6.3) is zero and concludes the proof. \square

Lemma 6.4. *The functions $\mathcal{U}_\alpha: \mathfrak{X}_{2N} \rightarrow \mathbb{R}_{>0}$ defined in equation (6.2) satisfy (COV) (1.2) with $\kappa = 4$.*

Proof. For any conformal map $\varphi: \mathbb{H} \rightarrow \mathbb{H}$, we have the identity $\frac{\varphi(z) - \varphi(w)}{z - w} = \sqrt{\varphi'(z)} \sqrt{\varphi'(w)}$ for all $z, w \in \overline{\mathbb{H}}$, see e.g. [KP16, Lemma 4.7]. Using this identity, we calculate

$$\frac{\mathcal{U}_\alpha(\varphi(x_1), \dots, \varphi(x_{2N}))}{\mathcal{U}_\alpha(x_1, \dots, x_{2N})} = \prod_{1 \leq i < j \leq 2N} \left(\frac{\varphi(x_j) - \varphi(x_i)}{x_j - x_i} \right)^{\frac{1}{2} \vartheta_\alpha(i, j)} = \prod_{1 \leq i < j \leq 2N} (\varphi'(x_j) \varphi'(x_i))^{\frac{1}{4} \vartheta_\alpha(i, j)}.$$

For each $j \in \{1, \dots, 2N\}$, the factor $\varphi'(x_j)$ comes with the total power $\frac{1}{4}(N(-1) + (N-1)(+1)) = -\frac{1}{4}$, which equals $h = (6 - \kappa)/2\kappa$ with $\kappa = 4$. Thus, \mathcal{U}_α satisfy (COV) (1.2) with $\kappa = 4$. \square

6.3 Combinatorics and a Binary Relation

In this section, we introduce combinatorial objects closely related to the link patterns $\alpha \in \text{LP}$, and present properties of them which are needed to complete the proof of Theorem 1.5. Results of this flavor appear in [KW11a, KW11b], and in [KKP17] for the context of pure partition functions. We follow the notations and conventions of the latter reference.

Dyck paths are walks on $\mathbb{Z}_{\geq 0}$ with steps of length one, starting and ending at zero. For $N \geq 1$, we denote the set of all Dyck paths of $2N$ steps by

$$\text{DP}_N := \left\{ \alpha: \{0, 1, \dots, 2N\} \rightarrow \mathbb{Z}_{\geq 0} \mid \alpha(0) = \alpha(2N) = 0, \text{ and } |\alpha(k) - \alpha(k-1)| = 1 \text{ for all } k \right\}.$$

To each link pattern $\alpha \in \text{LP}_N$, we associate a Dyck path, also denoted by $\alpha \in \text{DP}_N$, as follows. As in Equation (6.1), we write α as an ordered collection $\alpha = \{\{a_1, b_1\}, \dots, \{a_N, b_N\}\}$, with $a_1 < a_2 < \dots < a_N$ and $a_j < b_j$ for all $j \in \{1, \dots, N\}$. Then, we set $\alpha(0) = 0$ and for all $k \in \{1, \dots, 2N\}$,

$$\alpha(k) = \begin{cases} \alpha(k-1) + 1 & \text{if } k = a_r \text{ for some } r \\ \alpha(k-1) - 1 & \text{if } k = b_s \text{ for some } s. \end{cases}$$

Indeed, this defines a Dyck path $\alpha \in \text{DP}_N$. Conversely, for any Dyck path $\alpha: \{0, 1, \dots, 2N\} \rightarrow \mathbb{Z}_{\geq 0}$, we associate a link pattern α by associating to each up-step (i.e., step away from zero) an index a_r , for $r = 1, 2, \dots, N$, and to each down-step (i.e., step towards zero) an index b_s , for $s = 1, 2, \dots, N$, and setting $\alpha := \{\{a_1, b_1\}, \dots, \{a_N, b_N\}\}$. These two mappings $\text{LP}_N \rightarrow \text{DP}_N$ and $\text{DP}_N \rightarrow \text{LP}_N$ define a bijection between the sets of link patterns and Dyck paths, illustrated in Figure 6.1. We thus identify the elements α of these two sets and use the indistinguishable notation $\alpha \in \text{LP}_N$ and $\alpha \in \text{DP}_N$ for both.

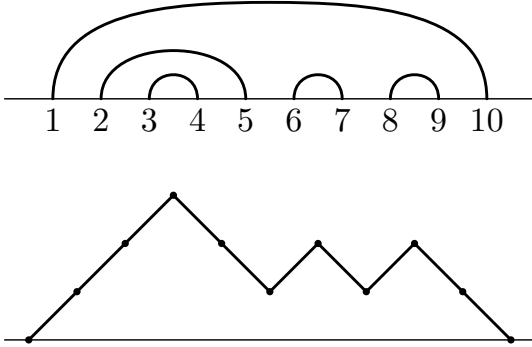


Figure 6.1: Illustration of the bijection $\text{LP}_N \leftrightarrow \text{DP}_N$ identifying link patterns and Dyck paths for $\alpha = \{\{1, 10\}, \{2, 5\}, \{3, 4\}, \{6, 7\}, \{8, 9\}\}$. Both the link pattern (top figure) and the Dyck path (bottom figure) are denoted by α .

These sets have a natural partial order \preceq measuring how nested their elements are: we define

$$\alpha \preceq \beta \quad \text{if and only if} \quad \alpha(k) \leq \beta(k) \text{ for all } k \in \{0, 1, \dots, N\}. \quad (6.6)$$

For instance, the rainbow link pattern $\underline{\mathbb{m}}_N$ is maximally nested — it is a maximal element in this partial order. In fact, the partial order \preceq is the transitive closure of a binary relation which was introduced by R. Kenyon and D. Wilson in [KW11a, KW11b] and K. Shigechi and P. Zinn-Justin in [SZ12]. We give a definition for this binary relation $\stackrel{\circ}{\leftarrow}$ that we have found the most suitable to the purposes of the present article. We refer to [KKP17, Section 2] for a detailed survey of this binary relation and many equivalent definitions of it; see also Figure 6.2 for an example. We define $\stackrel{\circ}{\leftarrow}$ as follows [KKP17, Lemma 2.5]:

Definition 6.5. Let $\alpha = \{\{a_1, b_1\}, \dots, \{a_N, b_N\}\} \in \text{LP}_N$ be ordered as in (6.1). Let $\beta \in \text{LP}_N$. Then, $\alpha \stackrel{\circ}{\leftarrow} \beta$ if and only if there exists a permutation $\sigma \in \mathfrak{S}_N$ such that

$$\beta = \{\{a_1, b_{\sigma(1)}\}, \dots, \{a_N, b_{\sigma(N)}\}\}.$$

For each $N \geq 1$, the incidence matrix \mathcal{M} of this relation on the set $\text{LP}_N \leftrightarrow \text{DP}_N$ is the $C_N \times C_N$ matrix $\mathcal{M} = (\mathcal{M}_{\alpha,\beta})$ whose matrix elements are

$$\mathcal{M}_{\alpha,\beta} = \mathbb{1}\{\alpha \stackrel{\circ}{\leftarrow} \beta\} = \begin{cases} 1 & \text{if } \alpha \stackrel{\circ}{\leftarrow} \beta \\ 0 & \text{otherwise.} \end{cases}$$

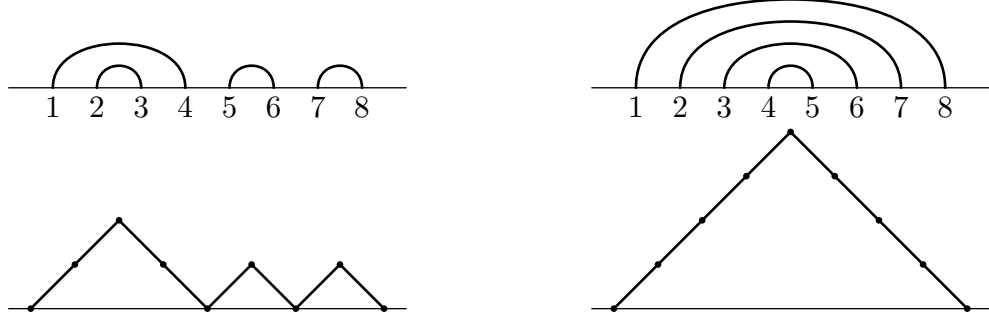


Figure 6.2: These two link patterns are comparable in the partial order \preceq , but uncomparable in the binary relation $\stackrel{\circ}{\leftarrow}$: the left link pattern is $\alpha = \{\{1, 4\}, \{2, 3\}, \{5, 6\}, \{7, 8\}\}$ and the right link pattern is $\beta = \{\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}\}$.

In order to complete the proof of Theorem 1.5, we need to invert the matrix \mathcal{M} . For this purpose, we introduce some more combinatorial notions, related to skew-Young diagrams and their tilings. Let $\alpha \preceq \beta$. When the two Dyck paths $\alpha, \beta \in \text{DP}_N$ are drawn on the same coordinate system, their difference forms a (rotated) skew Young diagram, denoted by α/β , which can be thought of as a union of atomic squares — see Figure 6.3. We denote by $|\alpha/\beta|$ the number of atomic square tiles in the skew Young diagram α/β .

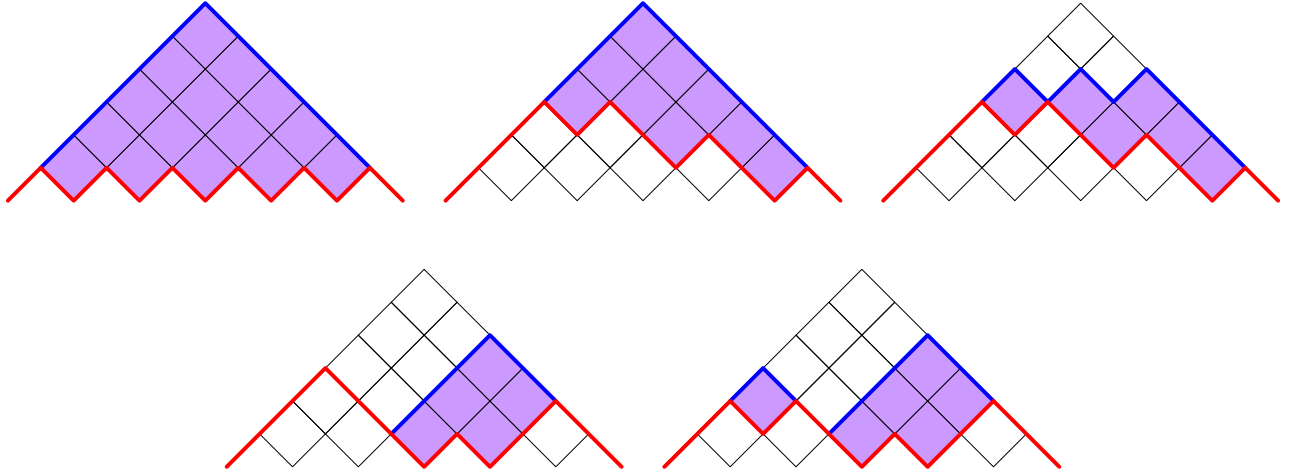


Figure 6.3: Skew Young diagrams α/β . The smaller Dyck path α (resp. larger β) is red (resp. blue).

Consider then tilings of the skew Young diagram α/β . The atomic square tiles form one possible tiling of α/β , a rather trivial one. In this article, following the terminology of [KW11a, KW11b, KKP17], we consider tilings of α/β by Dyck tiles, called Dyck tilings. A *Dyck tile* is a non-empty union of atomic squares, where the midpoints of the squares form a shifted Dyck path, see Figure 6.4. Note that also an atomic square is a Dyck tile. A *Dyck tiling* T of a skew Young diagram α/β is a collection of non-overlapping Dyck tiles whose union is $\bigcup T = \alpha/\beta$. Dyck tilings are also illustrated in Figure 6.4.

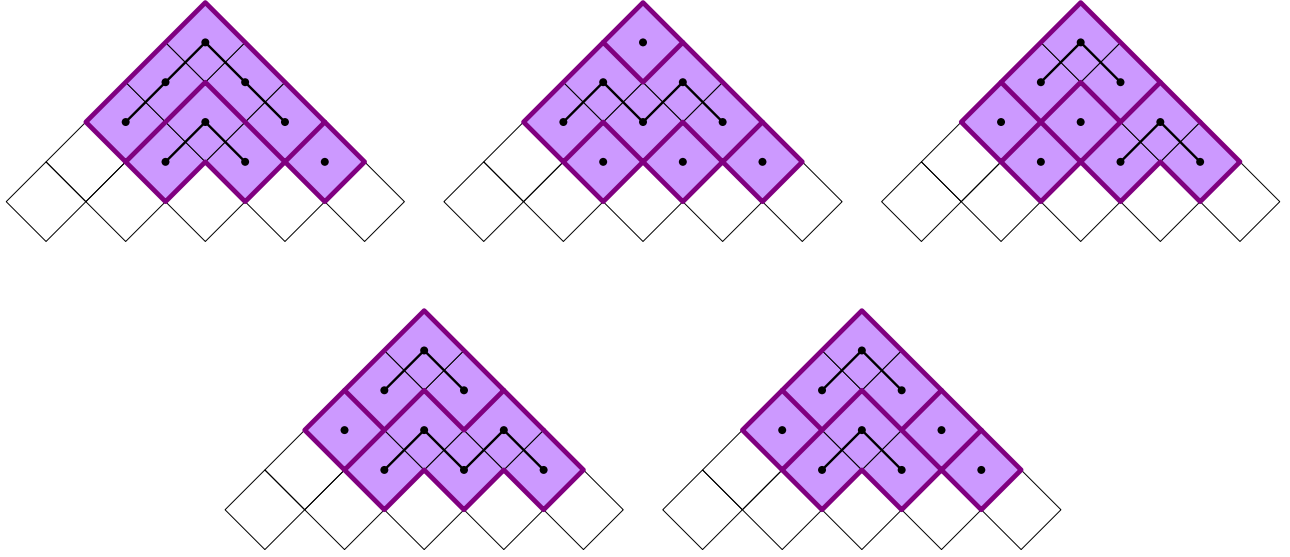


Figure 6.4: Examples of Dyck tilings, that is, tilings of a skew Young diagrams α/β by Dyck tiles.

The *placement* of a Dyck tile t is given by the integer coordinates (x_t, h_t) of the bottom left position of t , that is, the midpoint of the bottom left atomic square of t . If (x'_t, h_t) is the bottom right position of t , we call the closed interval $[x_t, x'_t] \subset \mathbb{R}$ the *horizontal extent* of t — see Figure 6.5 for an illustration.

A Dyck tile t_1 is said to *cover* a Dyck tile t_2 if t_1 contains an atomic square which is an upward vertical translation of some atomic square of t_2 . A Dyck tiling T of α/β is said to be *cover-inclusive* if for any two distinct tiles of T , either the horizontal extents are disjoint, or the tile that covers the other has horizontal extent contained in the horizontal extent of the other. See Figures 6.4 and 6.5 for illustrations.

After these preparations, we are now ready to recall from [KW11b, KKP17] the following result, which enables us to write an explicit formula for the pure partition functions for $\kappa = 4$ in Theorem 1.5.





Proposition 6.6. *The matrix \mathcal{M} is invertible with inverse given by*

$$\mathcal{M}_{\alpha, \beta}^{-1} = \begin{cases} (-1)^{|\alpha/\beta|} \# \mathcal{C}(\alpha/\beta) & \text{if } \alpha \preceq \beta \\ 0 & \text{otherwise,} \end{cases}$$

where $|\alpha/\beta|$ is the number of atomic square tiles in the skew Young diagram α/β and $\# \mathcal{C}(\alpha/\beta)$ denotes the number of cover-inclusive Dyck tilings of α/β , with the convention that $\# \mathcal{C}(\alpha/\alpha) = 1$.

Proof. This follows immediately from [KKP17, Theorem 2.9] with tile weight -1 . Originally, the proof appears in [KW11b, Theorems 1.5 and 1.6]. \square

The entries $\mathcal{M}_{\alpha, \beta}^{-1}$ are always integers, and the diagonal entries are all equal to one: $\mathcal{M}_{\alpha, \alpha}^{-1} = 1$ for all α . Thus, the formula (1.8) is lower-triangular in the partial order \succeq . For instance, we have $\mathcal{Z}_{\underline{\mathbb{M}}_N} = \mathcal{U}_{\underline{\mathbb{M}}_N}$ for the rainbow link pattern. We give in Tables 1 and 2 examples of the matrix \mathcal{M} and its inverse \mathcal{M}^{-1} .

LP _N with $N = 2$		
	1	0
	1	1





LP _N with $N = 2$		
	1	0
	-1	1

Table 1: The matrix elements of \mathcal{M} (left) and \mathcal{M}^{-1} (right) for $N = 2$.

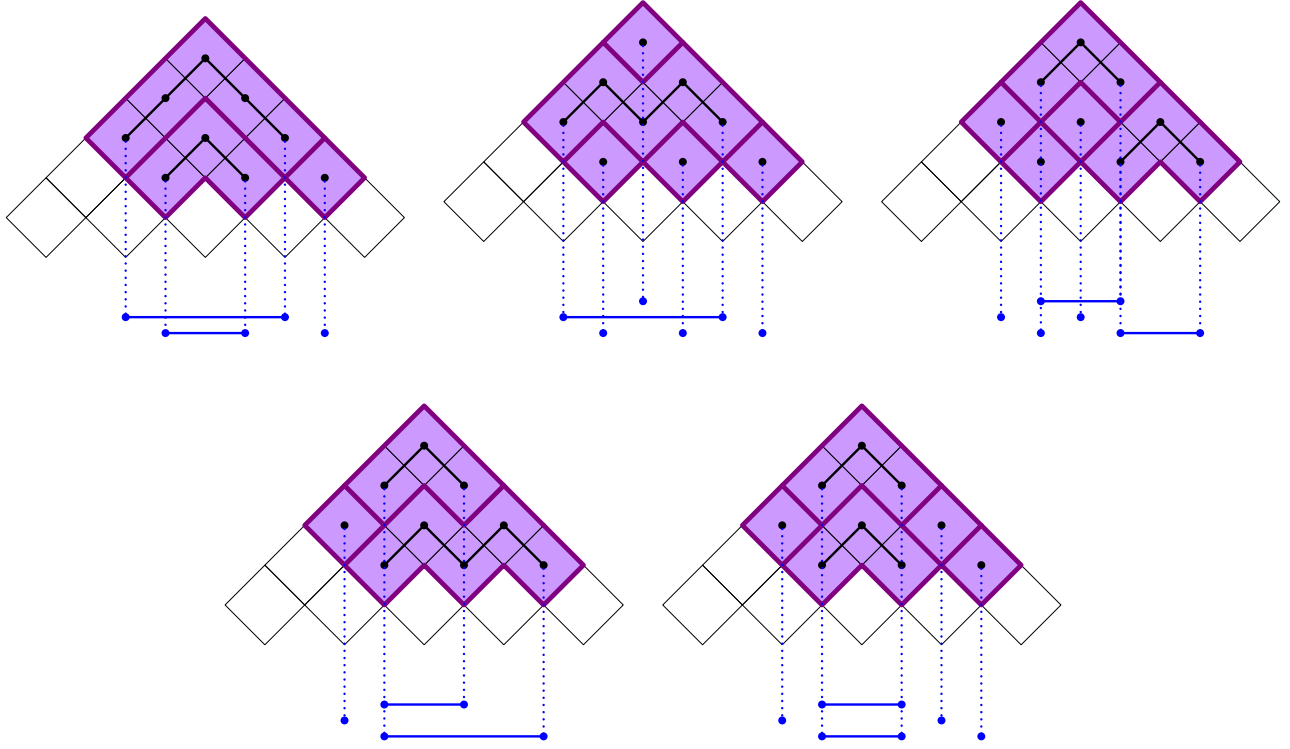


Figure 6.5: Examples of Dyck tilings with their horizontal extents illustrated. The two on the second row are cover-inclusive, but the three on the first row are not.

LP_N with $N = 3$					
	1	0	0	0	0
	1	1	0	0	0
	0	1	1	0	0
	0	1	0	1	0
	1	1	1	1	1

LP_N with $N = 3$					
	1	0	0	0	0
	-1	1	0	0	0
	1	-1	1	0	0
	1	-1	0	1	0
	-2	1	-1	-1	1

Table 2: The matrix elements of \mathcal{M} (top) and \mathcal{M}^{-1} (bottom) for $N = 3$.

To finish this preliminary section, we introduce notation for certain combinatorial operations on Dyck paths and summarize results about them that are needed to complete the proof of Theorem 1.5. In the bijection $LP_N \leftrightarrow DP_N$ illustrated in Figure 6.1, a link between j and $j + 1$ in $\alpha \in LP_N$ corresponds with an up-step followed by a down-step in the Dyck path α , so $\{j, j + 1\} \in \alpha$ is equivalent to j being a local maximum of the Dyck path $\alpha \in DP_N$. In this situation, we denote $\wedge^j \in \alpha$ and we say that α has an *up-wedge* at j . *Down-wedges* \vee_j are defined analogously, and an unspecified local extremum is called a

wedge \diamond_j . Otherwise, we say that α has a *slope* at j , denoted by $\times_j \in \alpha$. When α has a down-wedge, $\vee_j \in \alpha$, we define the *wedge-lifting operation* $\alpha \mapsto \alpha \uparrow \diamond_j$ by letting $\alpha \uparrow \diamond_j$ be the Dyck path obtained by converting the down-wedge \vee_j in α into an up-wedge \wedge^j .

We recall that, if a link pattern $\alpha \in \text{LP}_N$ has a link $\{j, j+1\} \in \alpha$, then we denote by $\alpha/\{j, j+1\} \in \text{LP}_{N-1}$ the link pattern obtained from α by removing the link $\{j, j+1\}$ and relabeling the remaining indices by $1, 2, \dots, 2N-2$ (see Figure 1.2). In terms of the Dyck path, this operation is called an up-wedge removal and denoted by $\alpha \setminus \wedge^j \in \text{DP}_{N-1}$. For Dyck paths, we can define a completely analogous down-wedge removal $\alpha \mapsto \alpha \setminus \vee_j$. Occasionally, it is not important to specify the type of wedge that is removed, so whenever α has either type of local extremum at j (that is, $\diamond_j \in \alpha$), we denote by $\alpha \setminus \diamond_j \in \text{DP}_{N-1}$ the two steps shorter Dyck path obtained by removing the two steps around \diamond_j , see Figure 6.6.

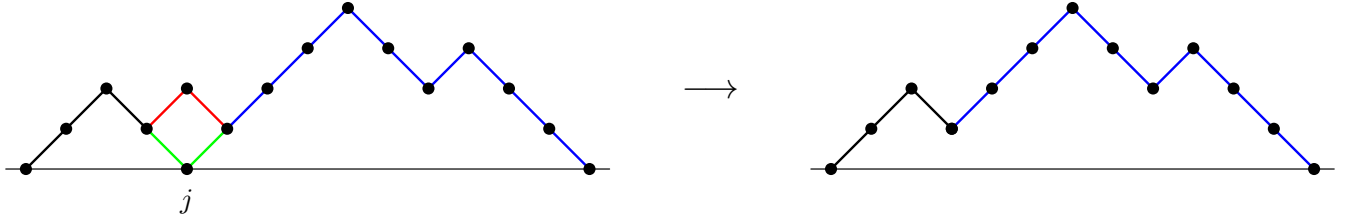


Figure 6.6: The removal of a wedge from a Dyck path. The left figure is the Dyck path $\alpha \in \text{DP}_N$ and the right figure the shorter Dyck path $\alpha \setminus \diamond_j \in \text{DP}_{N-1}$, with $j = 4$ and $N = 7$.

Now we are ready to collect results from [KKP17] that are needed later.

Lemma 6.7. *The following statements hold for Dyck paths $\alpha, \beta \in \text{DP}_N$.*

- (a): *Suppose $\wedge^j \notin \alpha$ and $\vee_j \in \beta$. Then, we have $\alpha \preceq \beta$ if and only if $\alpha \preceq \beta \uparrow \diamond_j$.*
- (b): *Suppose $\wedge^j \notin \alpha$. Then the Dyck paths $\beta \in \text{DP}_N$ such that $\beta \succeq \alpha$ and $\diamond_j \in \beta$ come in pairs, one containing an up-wedge and the other a down-wedge at j :*

$$\{\beta \in \text{DP}_N \mid \beta \succeq \alpha\} = \{\beta \mid \beta \succeq \alpha, \vee_j \in \beta\} \cup \{\beta \uparrow \diamond_j \mid \beta \succeq \alpha, \vee_j \in \beta\} \cup \{\beta \mid \beta \succeq \alpha, \times_j \in \beta\}.$$

- (c): *Suppose $\wedge^j \in \beta$. Then, we have $\alpha \xleftarrow{0} \beta$ if and only if $\diamond_j \in \alpha$ and $\alpha \setminus \diamond_j \xleftarrow{0} \beta \setminus \wedge^j$.*

- (d): *Suppose $\wedge^j \notin \alpha$, $\vee_j \in \beta$, and $\alpha \preceq \beta$. Then we have $\mathcal{M}_{\alpha, \beta}^{-1} = -\mathcal{M}_{\alpha, \beta \uparrow \diamond_j}^{-1}$.*

Proof. Parts (a) and (b) were proved e.g. in [KKP17, Lemma 2.11] (see also the remark below that lemma). Part (c) was proved e.g. in [KKP17, Lemma 2.12]. We give a short proof for Part (d). First, [KKP17, Lemma 2.15] says that if $\wedge^j \notin \alpha$, $\vee_j \in \beta$, and $\alpha \preceq \beta$, then we have $\#\mathcal{C}(\alpha/\beta) = \#\mathcal{C}(\alpha/(\beta \uparrow \diamond_j))$. On the other hand, Proposition 6.6 shows that $\mathcal{M}_{\alpha, \beta}^{-1} = (-1)^{|\alpha/\beta|} \#\mathcal{C}(\alpha/\beta)$. The claim follows from this and the observation that the number of Dyck tiles in a cover-inclusive Dyck tiling of $\alpha/(\beta \uparrow \diamond_j)$ is one more than the number of Dyck tiles in a cover-inclusive Dyck tiling of α/β , by [KKP17, proof of Lemma 2.15]. \square

6.4 Asymptotics and Proof of Lemma 6.9

To finish the proof of Theorem 1.5, we calculate the asymptotics of the conformal block functions \mathcal{U}_α . This proof is combinatorial, relying on results from [KW11a, KW11b, KKP17] discussed in Section 6.3. Recall that we identify any link pattern $\alpha \in \text{LP}_N$ with the corresponding a Dyck path $\alpha \in \text{DP}_N$.

Lemma 6.8. *The collection $\{\mathcal{U}_\alpha : \alpha \in \text{DP}\}$ of functions defined in (6.2) satisfy the asymptotics property*

$$\lim_{x_j, x_{j+1} \rightarrow \xi} \frac{\mathcal{U}_\alpha(x_1, \dots, x_{2N})}{(x_{j+1} - x_j)^{-1/2}} = \begin{cases} 0 & \text{if } \times_j \in \alpha \\ \mathcal{U}_{\alpha \setminus \wedge^j}(x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N}) & \text{if } \wedge^j \in \alpha \\ \mathcal{U}_{\alpha \setminus \vee_j}(x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N}) & \text{if } \vee_j \in \alpha, \end{cases} \quad (6.7)$$

for any $j \in \{1, \dots, 2N-1\}$ and $\xi \in (x_{j-1}, x_{j+2})$.

Proof. Fix $j \in \{1, \dots, 2N-1\}$. If $\times_j \in \alpha$, then either both j and $j+1$ are a -type indices with labels a_r, a_s , or both are b -type indices with labels b_r, b_s . In either case, we have $\vartheta_\alpha(j, j+1) = 1$, so the limit in (6.7) is zero. Assume then that $\wedge^j \in \alpha$ (resp. $\vee_j \in \alpha$). In this case, we have $j = b_s$ and $j+1 = a_r$ (resp. $j = a_r$ and $j+1 = b_s$) for some $r, s \in \{1, \dots, N\}$, so $\vartheta_\alpha(j, j+1) = -1$. By definition (6.2), we have

$$\begin{aligned} \mathcal{U}_\alpha(x_1, \dots, x_{2N}) &= \prod_{1 \leq k < l \leq 2N} (x_l - x_k)^{\frac{1}{2}\vartheta_\alpha(k, l)} \\ &= (x_{j+1} - x_j)^{-1/2} \prod_{\substack{k < l, \\ k, l \neq j, j+1}} (x_l - x_k)^{\frac{1}{2}\vartheta_\alpha(k, l)} \\ &\quad \times \prod_{k < j} (x_{j+1} - x_k)^{\frac{1}{2}\vartheta_\alpha(j+1, l)} (x_j - x_k)^{\frac{1}{2}\vartheta_\alpha(j, l)} \prod_{l > j+1} (x_l - x_{j+1})^{\frac{1}{2}\vartheta_\alpha(k, j+1)} (x_l - x_j)^{\frac{1}{2}\vartheta_\alpha(k, j)}. \end{aligned}$$

The first factor cancels with the normalization factor $(x_{j+1} - x_j)^{1/2}$ in the limit (6.7). The second product is independent of x_j, x_{j+1} and tends to $\mathcal{U}_{\alpha \setminus \wedge^j}(x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N})$ in the limit (6.7) (resp. tends to $\mathcal{U}_{\alpha \setminus \vee_j}$). Finally, the products in the last line tend to one in the limit (6.7), for we have $\vartheta_\alpha(k, j+1) = -\vartheta_\alpha(k, j)$ for all $k < j$ and $\vartheta_\alpha(j+1, l) = -\vartheta_\alpha(j, l)$ for all $l > j+1$. This finishes the proof. \square

Lemma 6.9. *The functions appearing on the right hand side of (1.8) satisfy (ASY) (1.3) with $\kappa = 4$.*

Proof. For convenience, denote the functions in question by

$$\tilde{\mathcal{Z}}_\alpha(x_1, \dots, x_{2N}) := \sum_{\beta \succeq \alpha} \mathcal{M}_{\alpha, \beta}^{-1} \mathcal{U}_\beta(x_1, \dots, x_{2N}). \quad (6.8)$$

Fix $j \in \{1, \dots, 2N-1\}$. For the asymptotics property (ASY) (1.3), we have two cases to consider: either $\{j, j+1\} \in \alpha$ or $\{j, j+1\} \notin \alpha$. As explained in Section 6.3, these can be equivalently written in terms of the Dyck path $\alpha \in \text{DP}_N$ as $\wedge^j \in \alpha$ and $\wedge^j \notin \alpha$. The asserted property (ASY) (1.3) with $\kappa = 4$ can thus be written in the following form: for all $\alpha \in \text{LP}_N$ and for all $j \in \{1, \dots, 2N-1\}$ and $\xi \in (x_{j-1}, x_{j+2})$,

$$\lim_{x_j, x_{j+1} \rightarrow \xi} \frac{\tilde{\mathcal{Z}}_\alpha(x_1, \dots, x_{2N})}{(x_{j+1} - x_j)^{-1/2}} = \begin{cases} 0 & \text{if } \wedge^j \notin \alpha \\ \tilde{\mathcal{Z}}_{\alpha \setminus \wedge^j}(x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N}) & \text{if } \wedge^j \in \alpha. \end{cases} \quad (6.9)$$

We prove the property (6.9) for $\tilde{\mathcal{Z}}_\alpha$ separately in the two cases $\wedge^j \in \alpha$ and $\wedge^j \notin \alpha$.

Assume first that $\wedge^j \notin \alpha$. We split the right hand side of (6.8) into three sums:

$$\tilde{\mathcal{Z}}_\alpha = \sum_{\beta \succeq \alpha : \vee_j \in \beta} \mathcal{M}_{\alpha, \beta}^{-1} \mathcal{U}_\beta + \sum_{\beta \succeq \alpha : \wedge^j \in \beta} \mathcal{M}_{\alpha, \beta}^{-1} \mathcal{U}_\beta + \sum_{\beta \succeq \alpha : \times_j \in \beta} \mathcal{M}_{\alpha, \beta}^{-1} \mathcal{U}_\beta.$$

Using Lemma 6.7(b), we combine the first and second sums to one sum over β such that $\vee_j \in \beta$, by replacing β in the second sum by $\beta \uparrow \diamond_j$. Furthermore, Lemma 6.7(d) shows that the coefficients in these two sums are related by $\mathcal{M}_{\alpha, \beta}^{-1} = -\mathcal{M}_{\alpha, \beta \uparrow \diamond_j}^{-1}$. Therefore, we obtain

$$\tilde{\mathcal{Z}}_\alpha = \sum_{\beta \succeq \alpha : \vee_j \in \beta} \mathcal{M}_{\alpha, \beta}^{-1} (\mathcal{U}_\beta - \mathcal{U}_{\beta \uparrow \diamond_j}) + \sum_{\beta \succeq \alpha : \times_j \in \beta} \mathcal{M}_{\alpha, \beta}^{-1} \mathcal{U}_\beta.$$

Now, it follows from Lemma 6.8 that the last sum vanishes in the limit (6.9), and that the functions \mathcal{U}_β and $\mathcal{U}_{\beta \uparrow \diamond_j}$ have the same limit, so they cancel. In conclusion, the limit (6.9) of $\tilde{\mathcal{Z}}_\alpha$ is zero when $\wedge^j \notin \alpha$.

Assume then that $\wedge^j \in \alpha$. By Proposition 6.6, the system (6.8) with $\alpha \in \text{DP}_N$ is invertible, and

$$\mathcal{U}_\beta(x_1, \dots, x_{2N}) = \sum_{\alpha \in \text{DP}_N} \mathcal{M}_{\beta, \alpha} \tilde{\mathcal{Z}}_\alpha(x_1, \dots, x_{2N}), \quad \text{for any } \beta \in \text{DP}_N, \quad (6.10)$$

where $\mathcal{M}_{\beta, \alpha} = \mathbb{1}\{\beta \stackrel{0}{\leftarrow} \alpha\}$. We already know by the first part of the proof that the limit (6.9) of $\tilde{\mathcal{Z}}_\alpha$ is zero when $\wedge^j \notin \alpha$. Therefore, taking the the limit (6.9) of the right hand side of (6.10) gives

$$\begin{aligned} & \sum_{\alpha \in \text{DP}_N} \mathbb{1}\{\beta \stackrel{0}{\leftarrow} \alpha\} \lim_{x_j, x_{j+1} \rightarrow \xi} \frac{\tilde{\mathcal{Z}}_\alpha(x_1, \dots, x_{2N})}{(x_{j+1} - x_j)^{-1/2}} \\ &= \sum_{\alpha : \wedge^j \in \alpha} \mathbb{1}\{\beta \stackrel{0}{\leftarrow} \alpha\} \lim_{x_j, x_{j+1} \rightarrow \xi} \frac{\tilde{\mathcal{Z}}_\alpha(x_1, \dots, x_{2N})}{(x_{j+1} - x_j)^{-1/2}} \quad \text{for any } \beta \in \text{DP}_N. \end{aligned} \quad (6.11)$$

We wish to calculate the limit in (6.11) for any fixed $\alpha \in \text{DP}_N$ such that $\wedge^j \in \alpha$.

By Lemma 6.7(c), we have $\beta \stackrel{0}{\leftarrow} \alpha$ if and only if $\diamond_j \in \beta$ and $\beta \setminus \diamond_j \stackrel{0}{\leftarrow} \alpha \setminus \wedge^j$. Now, choose $\beta \in \text{DP}_N$ such that $\wedge^j \in \beta$, and denote $\hat{\beta} = \beta \setminus \wedge^j$. Then, by Lemma 6.7(c), we have $\mathbb{1}\{\beta \stackrel{0}{\leftarrow} \alpha\} = \mathbb{1}\{\hat{\beta} \stackrel{0}{\leftarrow} \hat{\alpha}\}$ and we can re-index the sum in (6.11) by $\hat{\alpha} = \alpha \setminus \wedge^j$, to obtain

$$\sum_{\alpha \in \text{DP}_{N-1}} \mathbb{1}\{\hat{\beta} \stackrel{0}{\leftarrow} \hat{\alpha}\} \lim_{x_j, x_{j+1} \rightarrow \xi} \frac{\tilde{\mathcal{Z}}_\alpha(x_1, \dots, x_{2N})}{(x_{j+1} - x_j)^{-1/2}}. \quad (6.12)$$

On the other hand, with $\wedge^j \in \beta$, Lemma 6.8 gives the limit (6.9) of the left hand side of (6.10):

$$\begin{aligned} \frac{\mathcal{U}_\beta(x_1, \dots, x_{2N})}{(x_{j+1} - x_j)^{-1/2}} &= \mathcal{U}_{\hat{\beta}}(x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N}) \\ &= \sum_{\hat{\alpha} \in \text{DP}_{N-1}} \mathbb{1}\{\hat{\beta} \stackrel{0}{\leftarrow} \hat{\alpha}\} \tilde{\mathcal{Z}}_{\hat{\alpha}}(x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N}), \end{aligned} \quad (6.13)$$

where in the last equality we used (6.10) for $\hat{\beta} = \beta \setminus \wedge^j$. Combining (6.12) and (6.13), we arrive with

$$\begin{aligned} & \sum_{\hat{\alpha} \in \text{DP}_{N-1}} \mathcal{M}_{\hat{\beta}, \hat{\alpha}} \lim_{x_j, x_{j+1} \rightarrow \xi} \frac{\tilde{\mathcal{Z}}_\alpha(x_1, \dots, x_{2N})}{(x_{j+1} - x_j)^{-1/2}} \\ &= \sum_{\hat{\alpha} \in \text{DP}_{N-1}} \mathcal{M}_{\hat{\beta}, \hat{\alpha}} \tilde{\mathcal{Z}}_{\hat{\alpha}}(x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N}), \quad \text{for any } \hat{\beta} \in \text{DP}_{N-1}, \end{aligned} \quad (6.14)$$

where $\mathcal{M}_{\hat{\beta}, \hat{\alpha}} = \mathbb{1}\{\hat{\beta} \stackrel{0}{\leftarrow} \hat{\alpha}\}$ and $\alpha \in \text{LP}_N$ is determined by $\hat{\alpha} = \alpha \setminus \wedge^j$. Recalling that by Proposition 6.6, the system (6.14) is invertible, we can solve for the asserted limit (6.9). This concludes the proof. \square

We conclude with the following refined asymptotics property of the pure partition functions \mathcal{Z}_α , which was needed in Section 5. This property follows from the explicit expressions for the functions \mathcal{Z}_α , and the proof is independent of Section 5.

Corollary 6.10. *The collection $\{\mathcal{Z}_\alpha : \alpha \in \text{LP}\}$ of functions of Theorem 1.1 for $\kappa = 4$ satisfies the asymptotics property*

$$\lim_{\substack{\tilde{x}_j, \tilde{x}_{j+1} \rightarrow \xi, \\ \tilde{x}_i \rightarrow x_i \text{ for } i \neq j, j+1}} \frac{\mathcal{Z}_\alpha(\tilde{x}_1, \dots, \tilde{x}_{2N})}{(\tilde{x}_{j+1} - \tilde{x}_j)^{-1/2}} = \begin{cases} 0 & \text{if } \{j, j+1\} \notin \alpha \\ \mathcal{Z}_{\alpha/\{j, j+1\}}(x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N}) & \text{if } \{j, j+1\} \in \alpha \end{cases} \quad (6.15)$$

for any $j \in \{1, \dots, 2N-1\}$ and $x_1 < \dots < x_{j-1} < \xi < x_{j+2} < \dots < x_{2N}$.

Proof. Theorem 1.5 shows that $\mathcal{Z}_\alpha = \sum_\beta \mathcal{M}_{\alpha,\beta}^{-1} \mathcal{U}_\beta$, so (by the proof of Lemma 6.9), it suffices to note that \mathcal{U}_β satisfies

$$\lim_{\substack{\tilde{x}_j, \tilde{x}_{j+1} \rightarrow \xi, \\ \tilde{x}_i \rightarrow x_i \text{ for } i \neq j, j+1}} \frac{\mathcal{U}_\beta(\tilde{x}_1, \dots, \tilde{x}_{2N})}{(\tilde{x}_{j+1} - \tilde{x}_j)^{-1/2}} = \begin{cases} 0 & \text{if } \times_j \in \beta \\ \mathcal{U}_{\beta \setminus \wedge^j}(x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N}) & \text{if } \wedge^j \in \beta \\ \mathcal{U}_{\beta \setminus \vee^j}(x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N}) & \text{if } \vee_j \in \beta; \end{cases} \quad (6.16)$$

indeed, this is immediate from the explicit expression of \mathcal{U}_β in (6.2) and Lemma 6.8. \square

Remark 6.11. *In fact, the refined asymptotics (6.15) holds for all $\kappa \in (0, 4]$. This is a consequence of [FK15a, Lemma 4]. For $\kappa = 4$, however, Corollary 6.10 has an easier proof in our setting.*

6.5 Conformal Blocks for the GFF

In this final section, we give an interpretation for the functions \mathcal{U}_α appearing in Theorem 1.5 as partition functions associated to a particular boundary data of the GFF.

For $\alpha \in \text{LP}_N$, recall that we also denote by $\alpha \in \text{DP}_N$ the corresponding Dyck path. Let h_α be the GFF in \mathbb{H} with the following boundary data:

$$\lambda(2\alpha(k) - 1), \quad \text{if } x \in (x_k, x_{k+1}) \quad \text{for all } k \in \{0, 1, \dots, 2N\}. \quad (6.17)$$

Note that by this definition, the boundary value of h_α is always $-\lambda$ on $(-\infty, x_1) \cup (x_{2N}, \infty)$, and $+\lambda$ on $(x_1, x_2) \cup (x_{2N-1}, x_{2N})$. Define

$$\mathcal{H}_\alpha(k) := \lambda(\alpha(k-1) + \alpha(k) - 1) \quad \text{for all } k \in \{1, 2, \dots, 2N\}. \quad (6.18)$$

Then we always have $\mathcal{H}_\alpha(1) = \mathcal{H}_\alpha(2N) = 0$.

Proposition 6.12. *Fix $\alpha \in \text{LP}_N$. Let h_α be the GFF in \mathbb{H} with boundary data given by (6.17). For $j \in \{1, \dots, N\}$, let η_{a_j} be the level line of h_α with height $\mathcal{H}_\alpha(a_j)$ and η_{b_j} the level line of $-h_\alpha$ with height $-\mathcal{H}_\alpha(b_j)$. Then, the collection $(\eta_1, \dots, \eta_{2N})$ is a local N -SLE₄ with partition function \mathcal{U}_α .*

Proof. Clearly, the collection $(\eta_1, \dots, \eta_{2N})$ satisfies the conformal invariance (CI) and the domain Markov property (DMP) appearing in the definition of local multiple SLEs in Section 4.3. Thus, we only need to check the marginal law property (MARG) for each curve. We show this for η_{a_j} for $j \in \{1, \dots, N\}$; the proof for η_{b_j} is similar. On the one hand, since η_{a_j} is the level line of h_α with height $\mathcal{H}_\alpha(a_j)$, its marginal law is SLE₄(ρ) with force points $\{x_1, \dots, x_{2N}\} \setminus \{x_{a_j}\}$, where each x_{a_k} is a force point with weight $+2$ and each x_{b_k} is a force point with weight -2 . Therefore, the driving function W_t of η_{a_j} satisfies

$$\begin{aligned} dW_t &= 2dB_t + \sum_{k \neq a_j} \frac{\rho_k dt}{W_t - V_t^k}, \quad \text{where } \rho_k = \begin{cases} +2, & \text{if } k \in \{a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_N\}, \\ -2, & \text{if } k \in \{b_1, \dots, b_N\}. \end{cases} \\ dV_t^k &= \frac{2dt}{V_t^k - W_t}, \quad \text{for } k \neq a_j, \end{aligned} \quad (6.19)$$

where V_t^k are the time evolutions of the force points x_k , for $k \neq a_j$. On the other hand, (6.19) coincides with the SDE system (4.11) of (MARG) with $F_j = \partial_{a_j} \mathcal{U}_\alpha / \mathcal{U}_\alpha$, since by definition (6.2) of \mathcal{U}_α , we have

$$4 \frac{\partial_{a_j} \mathcal{U}_\alpha}{\mathcal{U}_\alpha} = \sum_{k \neq a_j} \frac{2\vartheta_\alpha(k, a_j)}{W_t - V_t^k} = \sum_{k \neq a_j} \frac{\rho_k}{W_t - V_t^k}.$$

This completes the proof. \square

Let $\alpha = \sqcap \sqcap_N$ be the completely unnested link pattern. Then $h_{\sqcap \sqcap_N}$ is a GFF in \mathbb{H} with alternating boundary conditions, $\mathcal{H}_{\sqcap \sqcap_N}(k) = 0$ for all k , and $\mathcal{U}_{\sqcap \sqcap_N} = \mathcal{Z}_{\text{GFF}}^{(N)}$. This is the situation discussed in Section 5.4. By Theorem 1.4, all connectivities $\beta \in \text{LP}_N$ for the level lines of $h_{\sqcap \sqcap_N}$ have a positive chance. However, for a general link pattern $\alpha \in \text{LP}_N \setminus \{\sqcap \sqcap_N\}$, the boundary conditions for h_α are more complicated, and its level lines cannot necessarily form all of the different connectivities: only level lines of h_α and level lines of $-h_\alpha$ with respective heights \mathcal{H} and $-\mathcal{H}$ of the same magnitude can connect with each other. For example, when $\alpha = \underline{\sqcap}_N$, then $h_{\underline{\sqcap}_N}$ is a GFF with the following boundary data:

$$\begin{cases} \lambda(2j-1), & \text{if } x \in (x_j, x_{j+1}), \quad \text{for all } j \in \{0, 1, \dots, N\}, \\ \lambda(4N-1-2j), & \text{if } x \in (x_j, x_{j+1}), \quad \text{for all } j \in \{N+1, N+2, \dots, 2N\}, \end{cases}$$

and the heights of the level lines are $\mathcal{H}_{\underline{\sqcap}_N}(k) = 2\lambda(k-1)$ for $k \in \{1, \dots, N\}$ and $\mathcal{H}_{\underline{\sqcap}_N}(k) = 2\lambda(2N-k)$ for $k \in \{N+1, \dots, 2N\}$. In this case, we have $\mathcal{U}_{\underline{\sqcap}_N} = \mathcal{Z}_{\underline{\sqcap}_N}$, and for $j \in \{1, \dots, N\}$, the curve η_j merges with η_{2N+1-j} almost surely, that is, the level lines necessarily form the rainbow connectivity $\underline{\sqcap}_N$. The marginal law of η_1 is $\text{SLE}_4(+2, \dots, +2, -2, \dots, -2)$ in \mathbb{H} from x_1 to x_{2N} with force points (x_2, \dots, x_{2N-1}) , where x_k (resp. x_l) is a force point with weight $+2$ for $k \leq N$ (resp. with weight -2 for $k \geq N+1$).

Remark 6.13. For each link pattern $\alpha \in \text{LP}_N$, we associate a balanced subset $S(\alpha) \subset \{1, 2, \dots, 2N\}$ (that is, a subset containing equally many even and odd indices) as follows. Write $\alpha = \{\{a_1, b_1\}, \dots, \{a_N, b_N\}\}$ as an ordered collection as in (6.1). Define

$$S(\alpha) := \{a_r \mid r \in \{1, 2, \dots, N\} \text{ and } a_r \text{ is odd}\} \cup \{b_s \mid s \in \{1, 2, \dots, N\} \text{ and } b_s \text{ is even}\}.$$

Let h be the GFF in \mathbb{H} with alternating boundary conditions. Then the probability that the level lines of h connect the points with indices in $S(\alpha)$ among themselves and the points with indices in the complement $\{1, 2, \dots, 2N\} \setminus S(\alpha)$ among themselves equals

$$\frac{\mathcal{U}_\alpha(x_1, \dots, x_{2N})}{\mathcal{Z}_{\text{GFF}}^{(N)}(x_1, \dots, x_{2N})}.$$

This fact was proved in [KW11a] for interfaces in the double-dimer model. The corresponding claim for the level lines of the GFF can be proved similarly.

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